이학 박사 학위논문

Curvature flows with a flat side

(평평한 측면이 있는 곡률 흐름)

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이 논문을 이학 박사 학위논문으로 제출함

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Curvature flows with a flat side

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Dedicated to my wife Yeonhee

Abstract

Curvature flows with a flat side

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We study the near-the-interface behavior of a compact convex scalar curvature flow with a flat side. Under suitable initial conditions on the flat side, we show that the interface propagates with a finite and non-degenerate speed and the level set with a finite speed, until the flat side vanishes. Then we get optimal derivative estimates of the pressure-like function, non-degeneracy of the speed of the level set, optimal decay estimates of curvatures near the interface, and a generalized version of Kim-Lee-Rhee's curvature lower bound, from which we obtain the Hölder regularity of the ratio of the curvature to the optimal decay rate up to the free boundary. In the end, we demonstrate the all-time existence of a solution which is smooth up to the interface in its support.

Key words: 35A01 Existence problems for PDEs, 35K65 Degenerate parabolic equations, 35R35 Free boundary problems for PDEs, 53C44 Geometric evolution equations

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Chapter 1

Introduction

1.1 The problem of the scalar curvature flow

This paper concerns the regularity of the free boundary problem associated with the scalar curvature flow with a flat side. We begin with the evolution equation governing the scalar curvature flow. Let a compact hypersurface Σ in the $(n+1)$-dimensional Euclidean space be given. We assume $n \geq 3$. Suppose that the body evolves in time by an embedding $X : \Sigma \times [0, T) \to \mathbb{R}^{n+1}$. We denote the image of X at time t by M_t. Suppose that the evolution occurs in the inward normal direction $N = -\nu$ at each point x of the hypersurface and that the speed is given by the scalar curvature σ_2. Then we have the following evolution equation:

$$\frac{\partial X}{\partial t}(x,t) = -\sigma_2(x,t)\nu(x,t) \text{ with } X(x,0) = X_0(x). \tag{1.1.1}$$

A *scalar curvature flow* is a solution to the equation above.

1.2 The history of the research on flows with a flat side

W. Firey [9] first considered the evolution of the Gauss curvature flow of compact surfaces. Hamilton [10] showed that if a Gauss curvature flow initially contains a flat side, then there will be a smaller flat side a little later and it takes some time for the surface to become strictly convex.

P. Daskalopoulos with R. Hamilton [4] studied the solvability and regularity of the interface Γ between the Gauss curvature flow and its flat side, by viewing the flow as a free boundary problem. They showed that the solution exists and is smooth up the interface in its support, for a short time.

P. Daskalopoulos with K.-A. Lee [6] showed the existence of regular solutions to a certain degenerate parabolic equation of the non-divergence form. Using these results of [6] for the model equation under certain coordinates, they [7] showed that the solution exists smoothly and the interface is smooth for all time until the flat side vanishes.

On the other hand, P. Daskalopoulos with R. Hamilton [3] studied the n-dimensional porous medium equation with a flat side. They showed the C^∞ regularity of the free boundary for a short time, using the regularity of a model degenerate equation which is obtained by changing coordinates.

P. Daskalopoulos , R. Hamilton, and K.-A. Lee showed [5] that the square root of the pressure is kept concave by the porous medium equation and that the weak solution is smooth up to the interface in its support and its free boundary is smooth for all time.

K.-A. Lee and E. Rhee [14] [15] considered the evolution of a rotation-invariant surface with a concave side and showed that a smooth solution exists up to the free boundary and for a long time while the free boundary is smooth. L. Kim and K.-A. Lee showed the same result for the α-th power of Gauss curvature flows in [12].

1.3 The equation of the flow in the local coordinates

We need to find the optimal regularity of the hypersurface near the free boundary where the curvatures become degenerate. Let us assume that the embedded hypersurface of the scalar curvature flow is given as the graph of a smooth function $y = f(x(t), t)$ for $x \in \mathbb{R}^n$. Then the evolution of the graph under the scalar curvature flow is given by

$$\frac{\partial f}{\partial t} = \sigma_2 \sqrt{1 + |\nabla_x f|^2}. \tag{1.3.1}$$

Now using $\{x_i\} \in \mathbb{R}^n$ as a local coordinate chart, we evaluate the scalar curvature from the metric g_{ij}, the second fundamental form h_{ij}, and the Weingarten map h^i_j, as in Ecker [8]. We have

$$\begin{aligned} \frac{\partial X}{\partial x^i} &= \left(\overrightarrow{e}_i, \frac{\partial f}{\partial x^i}\right), \ \frac{\partial^2 X}{\partial x^i x^j} = \left(0, \frac{\partial^2 f}{\partial x^i \partial x^j}\right), \ 1 \le i, j \le n, \\ g_{ij} &= \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^i} \right\rangle = \delta_{ij} + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}, \ g^{ij} = \left(\delta_{ij} - \frac{\partial_i f \partial_j f}{1 + |\nabla_x f|^2}\right), \ \ 1 \le i, j \le n, \end{aligned} \tag{1.3.2}$$

Since the normal vector $-\overrightarrow{\nu} = \overrightarrow{n} = \frac{(-\nabla_x f, 1)}{\sqrt{1+|\nabla_x f|^2}}$, we have

$$\begin{aligned} h_{ij} &= \left\langle -\overrightarrow{\nu}, \frac{\partial^2 X}{\partial x^i \partial x^j} \right\rangle = \frac{1}{\sqrt{1 + |\nabla_x f|^2}} \frac{\partial^2 f}{\partial x^i \partial x^j}, \ \ 1 \le i, j \le n, \\ h^i_j &= g^{ik} h_{kj} = \left(\delta_{ik} - \frac{\partial_i f \partial_k f}{1 + |\nabla_x f|^2}\right) \frac{1}{\sqrt{1 + |\nabla_x f|^2}} \frac{\partial^2 f}{\partial x^k \partial x^j}, \ \ 1 \le i, j \le n. \end{aligned} \tag{1.3.3}$$

Hence, the mean curvature, H, is given by $H(h^i_j) = \sum_{i=1}^n h^i_i = \sum_{i,j=1}^n g^{ij} h_{ij}$ and equal to

$$H = \sum_{i,j=1}^n \left(\delta_{ij} - \frac{\partial_i f \partial_j f}{1 + |\nabla_x f|^2}\right) \frac{1}{\sqrt{1 + |\nabla_x f|^2}} \frac{\partial^2 f}{\partial x^i \partial x^j}, \tag{1.3.4}$$

and the square sum $|A|^2 = \sum_{i,j=1}^n h^i_j h^j_i = \sum_{i,j=1}^n g^{ik} h_{kj} g^{jl} h_{li}$ of the principal curvatures is

$$|A|^2 = \left(\delta_{ik} - \frac{\partial_i f \partial_k f}{1 + |\nabla_x f|^2}\right) \frac{1}{\sqrt{1 + |\nabla_x f|^2}} \frac{\partial^2 f}{\partial x^k \partial x^j} \left(\delta_{jl} - \frac{\partial_j f \partial_l f}{1 + |\nabla_x f|^2}\right) \frac{1}{\sqrt{1 + |\nabla_x f|^2}} \frac{\partial^2 f}{\partial x^l \partial x^i}, \tag{1.3.5}$$

so that the scalar curvature $\sigma_2 = \sigma_2(h^i_j) = \frac{1}{2}(H^2 - |A|^2) = \frac{1}{2}\sum_{i,j,k,l=1}^n g^{ik}g^{jl}(h_{ki}h_{lj} - h_{kj}h_{li})$ is

$$\sigma_2 = \frac{1}{2(1+|\nabla_x f|^2)}\sum_{i,j,k,l=1}^n \Big(\delta_{ik} - \frac{\partial_i f \partial_k f}{1+|\nabla_x f|^2}\Big)\Big(\delta_{jl} - \frac{\partial_j f \partial_l f}{1+|\nabla_x f|^2}\Big)\Big(\frac{\partial^2 f}{\partial x^k \partial x^i}\frac{\partial^2 f}{\partial x^l \partial x^j} - \frac{\partial^2 f}{\partial x^k \partial x^j}\frac{\partial^2 f}{\partial x^l \partial x^i}\Big). \tag{1.3.6}$$

From the formula of the scalar curvature (1.3.6), we get the following lemma.

Lemma 1.3.1. *The scalar curvature flow is given by the graph of the function f, solving*

$$f_t = \frac{1}{2\sqrt{1+|\nabla_x f|^2}}\sum_{i,j,k,l=1}^n \Big(\delta_{ik} - \frac{f_i f_k}{1+|\nabla_x f|^2}\Big)\Big(\delta_{jl} - \frac{f_j f_l}{1+|\nabla_x f|^2}\Big)(f_{ki}f_{lj} - f_{kj}f_{li}). \tag{1.3.7}$$

Let us define $g = \sqrt{2f}$ and call it the pressure-like function. Let $I = 1 + g^2|\nabla_x g|^2$. Then we can express its evolution equation as the following:

$$\begin{aligned} g_t =& \frac{1}{2\sqrt{I}}\sum_{i,j=1}^n \Big(g(g_{ii}g_{jj} - g_{ij}^2) + g_j^2 g_{ii} + g_i^2 g_{jj} - 2g_i g_j g_{ij}\Big) \\ &- \frac{1}{I^{3/2}}g^2\sum_{i,j,k=1}^n g_i g_k\Big(g(g_{ik}g_{jj} - g_{jk}g_{ij}) + g_j^2 g_{ik} + g_i g_k g_{jj} - g_i g_j g_{jk} - g_j g_k g_{ij}\Big). \end{aligned} \tag{1.3.8}$$

Let $\Gamma_\varepsilon(t)$ be the level set $\{(x, g(x,t))|g = \varepsilon\}$. Seeing $\Gamma_\varepsilon(t)$ as a hypersurface in $\mathbb{R}^n$, we can take $\nu = e_1 \in \mathbb{R}^n$ to be its outer normal vector in $\mathbb{R}^n$. If we choose an arbitrary point (x,t) with its point on the graph $(x, g(x,t)) \in \Gamma_\varepsilon(t)$, then the equation 1.3.8 at (x,t) becomes

$$g_t(x,t) = \frac{1}{2\sqrt{1+g^2g_1^2}}\sum_{i,j=2}^n g(g_{ii}g_{jj} - g_{ij}^2) + \frac{1}{(1+g^2g_1^2)^{3/2}}\sum_{i=2}^n \Big(g(g_{ii}g_{11} - g_{i1}^2) + g_1^2 g_{ii}\Big), \tag{1.3.9}$$

In particular, on the interface $\Gamma(t)$ of the flat side where $g = 0$, we have $g_t = g_\nu^2 \Delta_\tau g = g_\nu^3 H_\Gamma$ where H_Γ is the mean curvature of the interface $\Gamma(t)$. Likewise, we denote by H_{Γ_ε} the mean curvature of the level set $\Gamma_\varepsilon(t)$ for $\varepsilon > 0$.

1.4 Our assumptions on the flat side and along the interface

In this work, we impose the same assumptions as in P. Daskalopoulos and K.-A. Lee [7], the two-dimensional case. Specifically, we assume the following conditions:

(A1) The hypersurface Σ at time $t = 0$, which is compact, satisfies

$$\Sigma = \Sigma_0 \cup \overline{\Sigma}_1$$

where Σ_0 is the flat side and $\overline{\Sigma}_1$ is the strictly convex side of the hypersurface. The interface between the two parts is

$$\Gamma = \Sigma_0 \cap \overline{\Sigma}_1.$$

(A2) Because the equation (1.3.7) is invariant under both rotation and translation, we can assume that Σ_0 is in the hyperplane $\{x_{n+1} = 0\}$ and Σ_1 lies above the hyperplane $\{x_{n+1} = 0\}$.

(A3) The lower part Σ_1 of $\overline{\Sigma}_1$ is parametrized by a function f, i.e. Σ_1 is the graph of a function

$$x_{n+1} = f(x)$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ containing Σ_0. We assume that Ω is contained in an n-dimensional ball B_R for some $R > 0$, f is differentiable on Ω, and $\Omega = \{x \in \mathbb{R}^n : |Df(x)| < \infty\}$.

(A4) The function f vanishes quadratically at Γ, in other words we assume that at time $t = 0$, there is some number $\lambda > 0$ such that for any $x \in \Gamma$ and for any tangential direction τ at Γ,

$$|Dg(x)| \geq \lambda \;\text{ and }\; D^2_{\tau\tau}g(x) \geq \lambda \tag{1.4.1}$$

and both $|Dg(x)|$ and $D^2_{\tau\tau}g(x)$ are finite. This non-degeneracy condition makes Γ start moving at any point at time $t = 0$ and the flow behaves as a free boundary problem.

(A5) A closed disc $D_{\rho_0} = \{x \in \mathbb{R}^n; |x| \leq \rho_0\}$ is contained in the flat side $\Sigma_0(t)$, whose area should be nonzero for time $0 \leq t \leq T$ and $0 < T < T_c$, where T_c is the time when the area of the the flat side shrinks to zero.

(A6) Throughout the paper, we also assume, without loss of generality, that

$$\inf_{x \in \partial\Omega(t)} f(x,t) \geq 3, \; 0 \leq t \leq T_c \tag{1.4.2}$$

where $\Omega(t) = \{x \in \mathbb{R}^n : |Df(x,t)| < \infty\}$.

(A7) The initial hypersurface has the regularity that the transformed function h (1.6.1) of $g = \sqrt{2f}$ is of class $C^{2,\beta}_s$ (1.6.3) for the metric (1.6.2), up to the interface $x_{n+1} = 0$ for some $0 < \beta < 1$.

1.5 Existence of strictly convex scalar curvature flows

Kim and Lee proved in [11] that if the initial compact hypersurface is strictly convex, then a smooth solution to scalar curvature flow exists and remains strictly convex, for all time before the flow shrinks to a point. This implies that

Lemma 1.5.1. *Under the assumption* (1.4.2)*, there exists a uniform constant* $M > 0$ *such that*

$$|\nabla f(x,t)| \leq Md(x, \Gamma(t)), \tag{1.5.1}$$

on $\{f(\cdot,t) \leq 2\}$ *for time* $0 \leq t \leq T_c$.

Proof. The principal curvatures on the part $\Sigma_1(t)$ are all bounded above by [11] for time $0 \leq t \leq T_c$. As $|\nabla f| = 0$ along $\Gamma(t)$, the assumption (A6) guarantees that there exists some positive constant M which does not depend on time t such that (1.5.1) holds on $\{f(\cdot,t) \leq 2\}$ for time $0 \leq t \leq T_c$. ◻

1.6 Short-time existence near the interface

Let us consider an arbitrary point on the interface $P \in \Gamma(t_0)$, for $t_0 \geq 0$ given. We assume that x_1 is the normal direction and $x_i,\ i = 2, \ldots, n$ are the tangential directions to $\Gamma(t_0)$ at P. Then we can use a local coordinate change

$$\Big(x_1, x_2 \ldots, x_n, g(x_1, x_2, \ldots, x_n, t)\Big) \rightarrow \Big(h(x_{n+1}, x_2, \ldots, x_n, t), x_2, \ldots, x_n, x_{n+1}\Big), \tag{1.6.1}$$

and consider the function h. More details about the transformed function h will be given in the section 5 later. Let $\mathcal{B}_\eta(P) := \{(x_{n+1}, (x_2, \ldots, x_n), t) = (z, y, t) | 0 \leq z \leq \eta^2, |y - y_0| \leq \eta, t_0 - \eta^2 \leq t \leq t_0\}$ near $P = (0, y_0, t_0)$ for $\eta > 0$. We define a metric $ds^2 := \frac{dx_{n+1}^2}{x_{n+1}} + \sum_{i=2}^n dx_i^2$ with a distance function

$$s((x_1, t_1), (x_2, t_2)) := |\sqrt{x_{1,n+1}} - \sqrt{x_{2,n+1}}| + |(x_{1,2}, \ldots, x_{1,n}) - (x_{2,2}, \ldots, x_{2,n})| + \sqrt{|t_1 - t_2|}, \tag{1.6.2}$$

and the space of functions $C_s^{2,\alpha}(\mathcal{B}_\eta(P))$ with respect to the Riemannian metric ds^2 by

$$C_s^{2,\alpha}(\mathcal{B}_\eta(P)) := \{h \mid h, h_t, h_{n+1}, h_i, x_{n+1}h_{n+1,n+1}, \sqrt{x_{n+1}}h_{n+1,i}, h_{ij} \in C_s^\alpha(\mathcal{B}_\eta(P)), i, j = 2, \ldots n\}. \tag{1.6.3}$$

Now, we have the short-time existence of the solution h near a free boundary point $P \in \Gamma$.

Lemma 1.6.1. *Under all the assumptions in the subsection 1.4, there exists a solution $h \in C_s^{2+\beta}(\mathcal{B}_\eta(P))$ of the equation (5.1.3) in $\mathcal{B}_\eta(P)$ at any $P \in \Gamma$ for some uniform constant $\eta > 0$.*

Proof. At time $t_0 = 0$, the equation (5.1.3) and its linearization (5.1.4) satisfy the condition of Theorem 7.3 in [4] with $k = 0$ and minor changes in higher dimension $n \geq 3$, by (A4) and (A7). □

Thus, we get the short-time existence of the scalar curvature flow along the interface.

Theorem 1.6.2. *Under all the assumptions in the subsection 1.4, there exists a solution f of the equation* (1.3.7) *for $0 \leq t \leq T_0$ for some $T_0 > 0$ and it is smooth up to the interface in its support for $0 < t \leq T_0$.*

Proof. Let us cover the interface $\Gamma = \Gamma(0)$ by $\{\mathcal{B}_\eta(P) — P \in \Gamma\}$ for the uniform constant $\eta > 0$ in Lemma 1.6.1. Then the solution g of (1.3.8) exists and $g \in C_s^{2+\beta}$ in $\{0 \leq g \leq \varepsilon\}$, $0 \leq t \leq T_0$ for some $\varepsilon,\ T_0 > 0$. Since (1.3.8) and (3.1.5) satisfy the condition of Theorem 7.1 in [4], in $\{0 \leq g \leq \varepsilon\}, 0 \leq t \leq T_0$, g is smooth in $\{0 < g \leq \varepsilon\}, 0 < t \leq T_0$. As a smooth solution g exists in the part $\{g > \varepsilon/2\}$ inside the strictly convex side by [11], a solution $g = \sqrt{2f}$ exists for $0 \leq t \leq T_0$ and it is smooth up to the interface in its support, for $0 < t \leq T_0$. □

1.7 Main results

In this paper, we prove the following theorem:

Theorem 1.7.1. *Under all the assumptions in the subsection 1.4, the scalar curvature flow 1.1.1 admits a solution for all time $0 \leq t < T_c$. Moreover, the pressure-like function $g = \sqrt{2f}$ is smooth in $\Omega(t) = \{x \in \mathbb{R}^n; |Df(x,t)| < \infty\}$ up to the interface $\Gamma(t) = \Sigma_0(t) \cap \Sigma_1(t)$ on time $0 < t \leq T$ for all $T < T_c$. In particular the free boundary $\Gamma(t)$ between the strictly convex side and the flat side will be a smooth hypersurface in $\mathbb{R}^n$ for all time $0 < t < T_c$.*

1.8 Summary

The outline of this paper is as follows. In the section 2, we show that the interface Γ moves at a finite and non-degenerate speed, and the level set moves at a finite speed. In the section 3 we obtain the gradient estimate of the function g and the curvature estimates. In the section 5 we change the coordinates and get the Hölder regularity of the transformed function. Finally in the section 6, we prove the all-time existence and C^∞ regularity up to the interface, which is our goal.

1.9 Notations

Here are some notations which we will use throughout the paper:

- $\Gamma(t) := \Sigma_0(t) \cap \Sigma_1(t)$ where $\Sigma_0(t)$ is the flat side and $\Sigma_1(t)$ is the strictly convex side of the hypersurface at time t, which can be considered as the graph of a function $x_{n+1} = f(x)$ on $\Omega(t)$.
- $\Omega(t) := \{x \in \mathbb{R}^n : |Df(x,t)| < \infty\}$ and $\Omega_P(t) := \{x \in \mathbb{R}^n : g(x,t) \leq g(P,t)\}$ for $P \in \Omega(t)$.
- $\Gamma_\varepsilon(t) := \{x \in \mathbb{R}^n : g(x,t) = \varepsilon\}$ and $\Gamma_{f,\varepsilon}(t) := \{x \in \mathbb{R}^n : f(x,t) = \varepsilon\}$ for $\varepsilon > 0$. $\Gamma_0(t) := \Gamma(t)$.
- $\Gamma_P(t) := \partial\Omega_P(t) = g(P)$-level set of $g(\cdot, t)$ for $P \in \Omega(t)$.
- $\nu(P) \in \mathbb{R}^n$ is the exterior unit normal to $\Gamma_P(t)$ for $P \in \Omega(t)$.
- $\mathcal{A}_{\delta_0} := \{(x,t);\ 0 < f(x,t) \leq \delta_0,\ 0 \leq t \leq T\}$.
- $I := 1 + g^2|\nabla g|^2$ and $J := |\nabla g|^2 + g$.
- $R_{g,2} := \sum_{i,j=1}^n (g_{ii}g_{jj} - g_{ij}^2)$ and $\overline{R}_{g,2} := \sum_{i,j=2}^n (g_{ii}g_{jj} - g_{ij}^2) + \frac{2}{I}\sum_{i=2}^n (g_{ii}g_{11} - g_{1i}^2)$.
- $\mathcal{B}_\eta = \mathcal{B}_\eta(P) := \{(x_{n+1}, (x_2, \ldots, x_n), t) = (z, y, t) | 0 \leq z \leq \eta^2, |y - y_0| \leq \eta, t_0 - \eta^2 \leq t \leq t_0\}$ near $P = (0, y_0, t_0)$ for $\eta > 0$. $\mathcal{B}_r^\gamma = \mathcal{B}_r^\gamma(Q) := \{z \geq 0, |\sqrt{z} - \sqrt{z_0}| \leq r, \gamma|y - y_0'| \leq r, t_0' - r^2 \leq t \leq t_0'\}$ near $Q = (z_0, y_0', t_0')$ for $\gamma > 0, r > 0$. $C_\mu := \{z^2 + |y|^2 \leq \mu^2, -\mu^2 \leq t \leq 0\}$ for $\mu > 0$.

Chapter 2

Finite and non-degenerate speed of the interface of the flat side

This section deals with the speed of the interface of the flat side and the level sets. We show that the speed is finite and non-degenerate for the flat side, and finite for the level sets. Throughout this section, we assume that $g(x,t)$ is a solution of (1.3.8) and it is smooth up to the interface in its support for time $0 \le t \le T < T_c$. We begin with the following:

Lemma 2.0.1. *The scaled function with sufficiently small $\varepsilon > 0$*

$$f_\varepsilon(x,t) = \frac{1}{1+C\varepsilon} f((1+A\varepsilon)x,(1+B\varepsilon)t) \tag{2.0.1}$$

is a supersolution (or subsolution, respectively) of the equation (1.3.7) if $C \ge A$ and

$$\begin{aligned} B+C-4A \ &> \frac{5(C-A)|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2}, \ \textit{or} \\ B+C-4A \ &< \frac{(C-A)|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2}, \ \textit{respectively.} \end{aligned} \tag{2.0.2}$$

Proof. Let $A, B, C \in \mathbb{R}$ be constants. We scale the function $f(x,t)$ with those constants by

$$f_\varepsilon(x,t) = \frac{1}{1+C\varepsilon} f((1+A\varepsilon)x,(1+B\varepsilon)t)\ ,\ \varepsilon > 0\ . \tag{2.0.3}$$

Then its partial derivatives of the first order are

$$f_{\varepsilon,t}(x,t) = \frac{\partial f_\varepsilon}{\partial t} = \frac{1+B\varepsilon}{1+C\varepsilon} f_t((1+A\varepsilon)x,(1+B\varepsilon)t),$$

$$f_{\varepsilon,i} = \frac{\partial f_\varepsilon}{\partial x^i} = \frac{1+A\varepsilon}{1+C\varepsilon} f_i((1+A\varepsilon)x,(1+B\varepsilon)t),$$

with

$$|\nabla f_\varepsilon|^2 = \left(\frac{1+A\varepsilon}{1+C\varepsilon}\right)^2 |\nabla f((1+A\varepsilon)x,(1+B\varepsilon)t)|^2. \tag{2.0.4}$$

We denote $M = \left(\frac{1+C\varepsilon}{1+A\varepsilon}\right)^2$. Rearranging the equation of the scalar curvature flow about f_ε, we get the following equation:

$$\frac{\partial}{\partial_t} f_\varepsilon = \frac{E_{ijkl}(x)}{2\sqrt{1+|\nabla_x f_\varepsilon|^2}}(\delta_{ik} - \frac{f_{\varepsilon,i}f_{\varepsilon,k}}{1+|\nabla_x f_\varepsilon|^2})(\delta_{jl} - \frac{f_{\varepsilon,j}f_{\varepsilon,l}}{1+|\nabla_x f_\varepsilon|^2})(f_{\varepsilon,ki}f_{\varepsilon,lj} - f_{\varepsilon,kj}f_{\varepsilon,li}), \tag{2.0.5}$$

where

$$E_{ijkl}(x) = \frac{(1+B\varepsilon)(1+C\varepsilon)}{(1+A\varepsilon)^4}\left(\frac{1+|\nabla f_\varepsilon|^2}{1+M|\nabla f_\varepsilon|^2}\right)^{5/2} \frac{(1+M|\nabla f_\varepsilon|^2)\delta_{ik} - Mf_{\varepsilon,i}f_{\varepsilon,k}}{(1+|\nabla f_\varepsilon|^2)\delta_{ik} - f_{\varepsilon,i}f_{\varepsilon,k}} \frac{(1+M|\nabla f_\varepsilon|^2)\delta_{jl} - Mf_{\varepsilon,j}f_{\varepsilon,l}}{(1+|\nabla f_\varepsilon|^2)\delta_{jl} - f_{\varepsilon,j}f_{\varepsilon,l}}.$$

Since the right-hand side of the equation (2.0.5) is rotation invariant, we can take a coordinate system where the metric satisfies $g_{\varepsilon,ij} = \delta_{ij} + \frac{\partial f_\varepsilon}{\partial x^i}\frac{\partial f_\varepsilon}{\partial x^j} = 0$ and $\frac{\partial f_\varepsilon}{\partial x^i}\frac{\partial f_\varepsilon}{\partial x^j} = 0$ for $i \neq j$, and (2.0.5) becomes

$$\frac{\partial}{\partial_t} f_\varepsilon = \frac{E_{ijij}(x)}{2\sqrt{1+|\nabla_x f_\varepsilon|^2}}(1 - \frac{f_{\varepsilon,i}^2}{1+|\nabla_x f_\varepsilon|^2})(1 - \frac{f_{\varepsilon,j}^2}{1+|\nabla_x f_\varepsilon|^2})(f_{\varepsilon,ii}f_{\varepsilon,jj} - f_{\varepsilon,ij}^2), \tag{2.0.6}$$

where

$$1 - \frac{f_{\varepsilon,i}^2}{1+|\nabla_x f_\varepsilon|^2} = \frac{1+(|\nabla_x f_\varepsilon|^2 - f_{\varepsilon,i}^2)}{1+|\nabla_x f_\varepsilon|^2} \geq \frac{1}{1+|\nabla_x f_\varepsilon|^2} > 0,$$

and because the strictly convex side of the flow keeps being strictly convex for all time by [11],

$$f_{\varepsilon,ii}f_{\varepsilon,jj} - f_{\varepsilon,ij}^2 \geq 0.$$

Hence, f_ε becomes a supersolution (respectively, subsolution) if $E_{ijij} \geq 1$ (respectively, if $E_{ijij} \leq 1$) for every i, j, where

$$E_{ijij}(x) = \frac{(1+B\varepsilon)(1+C\varepsilon)}{(1+A\varepsilon)^4}\left(\frac{1+|\nabla f_\varepsilon|^2}{1+M|\nabla f_\varepsilon|^2}\right)^{5/2} \frac{1+M(|\nabla f_\varepsilon|^2 - f_{\varepsilon,i}^2)}{1+(|\nabla f_\varepsilon|^2 - f_{\varepsilon,i}^2)} \frac{1+M(|\nabla f_\varepsilon|^2 - f_{\varepsilon,j}^2)}{1+(|\nabla f_\varepsilon|^2 - f_{\varepsilon,j}^2)}.$$

As $\varepsilon \to 0^+$, we can use the first-order approximation $(1+C\varepsilon)^\alpha = 1 + \alpha C\varepsilon + O(\varepsilon^2)$:

$$\frac{(1+B\varepsilon)(1+C\varepsilon)}{(1+A\varepsilon)^4} = 1 + (B + C - 4A)\varepsilon + O(\varepsilon^2),$$

$$M = \left(\frac{1+C\varepsilon}{1+A\varepsilon}\right)^2 = (1+2C\varepsilon)(1-2A\varepsilon) + O(\varepsilon^2) = 1 + 2(C-A)\varepsilon + O(\varepsilon^2)\,,$$

$$\begin{aligned}\left(1+M|\nabla f_\varepsilon|^2\right) &= 1 + (1 + 2(C-A)\varepsilon + O(\varepsilon^2))|\nabla f_\varepsilon|^2 \\ &= (1+|\nabla f_\varepsilon|^2)(1 + (2(C-A)\varepsilon + O(\varepsilon^2))\frac{|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2})\,,\end{aligned}$$

$$\left(\frac{1+|\nabla f_\varepsilon|^2}{1+M|\nabla f_\varepsilon|^2}\right)^{5/2} = 1 - 5(C-A)\varepsilon\frac{|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2} + O(\varepsilon^2)\,,$$

$$
\begin{aligned}
&\left(1+M|\nabla f_\varepsilon|^2\right)\delta_{ik}-Mf_{\varepsilon,i}f_{\varepsilon,k}\\
&=(1+(1+2(C-A)\varepsilon)|\nabla f_\varepsilon|^2)\delta_{ik}-(1+2(C-A)\varepsilon)f_{\varepsilon,i}f_{\varepsilon,k}+O(\varepsilon^2)\\
&=((1+|\nabla f_\varepsilon|^2)\delta_{ik}-f_{\varepsilon,i}f_{\varepsilon,k})+2(C-A)\varepsilon(|\nabla f_\varepsilon|^2\delta_{ik}-f_{\varepsilon,i}f_{\varepsilon,k})+O(\varepsilon^2)\ ,
\end{aligned}
$$

$$
\frac{(1+M|\nabla f_\varepsilon|^2)\delta_{ik}-Mf_{\varepsilon,i}f_{\varepsilon,k}}{(1+|\nabla f_\varepsilon|^2)\delta_{ik}-f_{\varepsilon,i}f_{\varepsilon,k}}=1+2(C-A)\varepsilon\frac{|\nabla f_\varepsilon|^2\delta_{ik}-f_{\varepsilon,i}f_{\varepsilon,k}}{(1+|\nabla f_\varepsilon|^2)\delta_{ik}-f_{\varepsilon,i}f_{\varepsilon,k}}+O(\varepsilon^2),
$$

and hence, the factors $E_{ijkl}(x)$ satisfy, as $\varepsilon\to 0^+$,

$$
\begin{aligned}
E_{ijkl}(x)&=(1+(B+C-4A)\varepsilon)(1-5(C-A)\varepsilon\frac{|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2})\left(1+2(C-A)\varepsilon\frac{|\nabla f_\varepsilon|^2\delta_{ik}-f_{\varepsilon,i}f_{\varepsilon,k}}{(1+|\nabla f_\varepsilon|^2)\delta_{ik}-f_{\varepsilon,i}f_{\varepsilon,k}}\right)\\
&\cdot\left(1+2(C-A)\varepsilon\frac{|\nabla f_\varepsilon|^2\delta_{jl}-f_{\varepsilon,j}f_{\varepsilon,l}}{(1+|\nabla f_\varepsilon|^2)\delta_{jl}-f_{\varepsilon,j}f_{\varepsilon,l}}\right)+O(\varepsilon^2)\\
&=1+\varepsilon\left(B+C-4A-(C-A)\left(\frac{5|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2}-2\frac{|\nabla f_\varepsilon|^2\delta_{ik}-f_{\varepsilon,i}f_{\varepsilon,k}}{(1+|\nabla f_\varepsilon|^2)\delta_{ik}-f_{\varepsilon,i}f_{\varepsilon,k}}-2\frac{|\nabla f_\varepsilon|^2\delta_{jl}-f_{\varepsilon,j}f_{\varepsilon,l}}{(1+|\nabla f_\varepsilon|^2)\delta_{jl}-f_{\varepsilon,j}f_{\varepsilon,l}}\right)\right)\\
&+O(\varepsilon^2)
\end{aligned}
$$

and the factors $E_{ijij}(x)$ satisfy, as $\varepsilon\to 0^+$,

$$
\begin{aligned}
E_{ijij}(x)=&1+\varepsilon\left(B+C-4A-(C-A)\left(\frac{5|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2}-2\frac{|\nabla f_\varepsilon|^2-f_{\varepsilon,i}^2}{(1+|\nabla f_\varepsilon|^2)-f_{\varepsilon,i}^2}-2\frac{|\nabla f_\varepsilon|^2-f_{\varepsilon,j}^2}{(1+|\nabla f_\varepsilon|^2)-f_{\varepsilon,j}^2}\right)\right)\\
&+O(\varepsilon^2),\ \text{with}
\end{aligned}
$$

$$
\frac{|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2}\le\frac{5|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2}-2\frac{|\nabla f_\varepsilon|^2-f_{\varepsilon,i}^2}{(1+|\nabla f_\varepsilon|^2)-f_{\varepsilon,i}^2}-2\frac{|\nabla f_\varepsilon|^2-f_{\varepsilon,j}^2}{(1+|\nabla f_\varepsilon|^2)-f_{\varepsilon,j}^2}\le\frac{5|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2},\tag{2.0.7}
$$

implying that

$$E_{ijij}(x)\ge 1(\text{ respectively }E_{ijij}(x)\le 1)$$

if the inequality with $>$ (or $<$, respectively) in (2.0.2) holds for small $\varepsilon>0$. So the evolution equation (2.0.6) of the scaled function $f_\varepsilon(x,t)$ finishes the proof when $\varepsilon>0$ is sufficiently small. □

This implies the following lemma, from which we see that the propagating speed of the free boundary is finite.

Lemma 2.0.2. *There exist a constant $\delta_0>0$, a negative constant B and a positive constant C satisfying*

$$-Cf(x,t)+x\cdot\nabla f(x,t)+Btf_t(x,t)\ge 0\tag{2.0.8}$$

on the set $\mathcal{A}_{\delta_0}=\{(x,t);\ 0<f(x,t)\le\delta_0,\ 0\le t\le T\}$.

Proof. Because $f(x,t)$ is uniformly continuous on $t\in[0,T]$, for any $0<\eta<1$ there exists $0<\delta_0\ll\eta<1$ such that

$$\{x;0<f(x,t)\le\delta_0\}\subset\{x;d(x,\mathrm{Conv}(\Gamma(t)))\le\frac{\eta\rho_0}{2}\}\ ,\tag{2.0.9}$$

$$\{(1+\varepsilon)x; d(x, \text{Conv}(\Gamma(t))) \leq \frac{\eta\rho_0}{2}\} \subset \{x; d(x, \text{Conv}(\Gamma(t))) \leq \eta\rho_0\} \tag{2.0.10}$$

on $0 \leq t \leq T$, for all $\varepsilon \ll \delta_0$, where $\text{Conv}(\Gamma(t))$ is the convex hull of $\Gamma(t)$. Let $\Gamma(t) = \partial\{x; f(x,t) > 0\}$ be the interface. Consider the scaled function f_ε as in Lemma 2.0.1 with $A = 1$, $B = -\delta_0^2$, and $C = 8$. To prove this lemma, it is sufficient to show that $f_\varepsilon \geq f$ in $\mathcal{A}_{\delta_0}$ so that $\frac{d}{d\varepsilon}|_{\varepsilon=0} f_\varepsilon \geq 0$ in $\mathcal{A}_{\delta_0}$. The inequality condition (2.0.2) in Lemma 2.0.1 for f_ε to be a supersolution becomes

$$-\delta_0^2 + 4 > \frac{35|\nabla f_\varepsilon|^2}{1+|\nabla f_\varepsilon|^2}.$$

This inequality holds if $|\nabla f_\varepsilon|^2 \leq \frac{1}{12}$ and $0 < \delta_0 < 1$. By the equation (2.0.4), $|\nabla f_\varepsilon(x,t)| \leq |\nabla f((1+\varepsilon)x, (1+B\varepsilon)t)|$ with $A = 1$, $B = -\delta_0^2$, and $C = 8$ and it is enough to show that $|\nabla f((1+\varepsilon)x, (1-\delta_0^2\varepsilon)t)|^2 \leq \frac{1}{12}$.

Note that the function f is nondecreasing in time t and $f_t \geq 0$, because the scalar curvature σ_2 is nonnegative for all $t > 0$. The nonnegativity of σ_2 is derived from the fact that the strictly convex side of the scalar curvature flow remains strictly convex for all time, by [11].

Assume that $(x,t) \in \mathcal{A}_{\delta_0}$, i.e. $f(x,t) \leq \delta_0$. Then $f(x, (1-\delta_0^2\varepsilon)t) \leq \delta_0$ because $f_t \geq 0$. By (2.0.9), we have

$$d(x, \Gamma((1-\delta_0^2\varepsilon)t)) \leq \frac{\eta\rho_0}{2},$$

and by (2.0.10),

$$d((1+\varepsilon)x, \Gamma((1-\delta_0^2\varepsilon)t)) \leq \eta\rho_0$$

whose right-hand side becomes arbitrarily small if we choose η small. Consequently, by (1.5.1), we can make $|\nabla f((1+\varepsilon)x, (1-\delta_0^2\varepsilon)t)|^2 \leq \frac{1}{12}$.

By applying the comparison principle to the supersolutions f_ε and f, we can show that if additionally $f_\varepsilon \geq f$ on the parabolic boundary $\partial_p\mathcal{A}_{\delta_0} = \{(x,t); f(x,t) = \delta_0, 0 \leq t \leq T\} \cup \{x; f(x,0) \leq \delta_0,\}$ of $\mathcal{A}_{\delta_0}$, then $f_\varepsilon \geq f$ in $\mathcal{A}_{\delta_0}$. Now, the only remaining part is to prove that $f_\varepsilon \geq f$ on $\partial_p\mathcal{A}_{\delta_0}$. From the simple calculation

$$\begin{aligned}\frac{d}{d\varepsilon} f_\varepsilon(x,t) = &-\frac{C}{(1+C\varepsilon)^2} f((1+A\varepsilon)x, (1+B\varepsilon)t) + \frac{Ax}{1+C\varepsilon} \cdot \nabla_x f((1+A\varepsilon)x, (1+B\varepsilon)t) \\ &+ \frac{Bt}{1+C\varepsilon} f_t((1+A\varepsilon)x, (1+B\varepsilon)t),\end{aligned}$$

we have, for $A = 1$ in our assumption,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f_\varepsilon(x,0) = -Cf(x,0) + x \cdot \nabla_x f(x,0) = g\Big(-\frac{C}{2}g(x,0) + x \cdot \nabla_x g(x,0)\Big)$$

With the assumptions (A1), (A3) about strict convexity of f in its support, (A4) (1.4.1) and (A5) $|x| \geq \rho_0$ in the subsection 1.4, it holds that $|x \cdot \nabla_x g(x,0)| \geq C'$ for some uniform $C' > 0$ in $\mathcal{A}_{\delta_0}$. Thus, for small enough $\delta_0 > 0$, $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} f_\varepsilon(x,0) > 0$ on $\{x; 0 < f(x,0) \leq \delta_0\}$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} f_\varepsilon(x,0) = 0$ for $x \in \Gamma$, so that $f_\varepsilon(x,0) \geq f(x,0)$ for small $\varepsilon > 0$, unless x is in the interior of the flat side. When $f(x,t) = \delta_0$, we see that $d(x, \Gamma(t)) \leq \eta\rho_0/2 \leq \eta\rho_0$. Since f is convex in the radius r, the radial derivative f_r satisfies $f_r \geq \frac{\delta_0}{\eta\rho_0}$ and it holds that $r = d(0,x) \geq \rho_0$ and $x \cdot \nabla_x f = r f_r(x,t) \geq \rho_0 \frac{\delta_0}{\eta\rho_0} = \frac{\delta_0}{\eta}$ on $\partial_p \mathcal{A}_{\delta_0}$.

Hence, for $f = \delta_0$ on $\partial_p \mathcal{A}_{\delta_0}$, if η is small enough,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f_\varepsilon(x,t) = -Cf(x,t) + x \cdot \nabla_x f(x,t) + Btf_t(x,t) \geq -C\delta_0 + \frac{\delta_0}{\eta} - \delta_0^2 T|f_t|_{L^\infty} > 0,$$

which means that $f_\varepsilon \geq f$ on $\partial_p \mathcal{A}_{\delta_0}$, and the comparison principle finishes the proof. □

We can show that the free boundary moves with a finite speed when $0 \leq t \leq T$. The radius of a point x on the free boundary $\Gamma(t)$ is written by $r = \gamma(\theta, t)$ with $\theta = \frac{x}{|x|} \in \mathbb{S}^{n-1}$.

Lemma 2.0.3. *There is a constant $B < 0$ such that*

$$\gamma(\theta,t) \geq e^{-\frac{t-t_0}{|B|t_0}} \gamma(\theta, t_0) \tag{2.0.11}$$

for all $0 < t_0 \leq t \leq T < T_c$ and $\theta \in \mathbb{S}^{n-1}$.

Proof. From Lemma 2.0.2, for $0 < t_0 \leq t \leq T < T_c$, we get the inequality

$$0 \geq \frac{Cf(x,t)}{|B|t} - \frac{x}{|B|t} \cdot \nabla f(x,t) + f_t(x,t) \geq \frac{Cf(x,t)}{|B|T} - \frac{x}{|B|t_0} \cdot \nabla f(x,t) + f_t(x,t)$$

so that

$$\begin{aligned} &\frac{d}{dt}(e^{\frac{C}{|B|T}(t-t_0)} f(e^{-\frac{t-t_0}{|B|t_0}} x, t)) \\ &= e^{\frac{C}{|B|T}(t-t_0)} \left(\frac{Cf(e^{-\frac{t-t_0}{|B|t_0}} x, t)}{|B|T} - \frac{x}{|B|t_0} \cdot \nabla f(e^{-\frac{t-t_0}{|B|t_0}} x, t) + f_t(e^{-\frac{t-t_0}{|B|t_0}} x, t) \right) \leq 0 \end{aligned}$$

and hence for $x \in \Gamma(t_0)$

$$e^{\frac{C}{|B|T}(t-t_0)} f(e^{-\frac{t-t_0}{|B|t_0}} x, t) \leq f(x, t_0) = 0 \tag{2.0.12}$$

which implies the conclusion. □

For small $\varepsilon > 0$ the ε-level set of the function f moves with a finite speed as well, by the following theorem. Let us express the ε-level set by its radius $r = \gamma_\varepsilon(\theta, t)$.

Lemma 2.0.4. *There are constants $B < 0$, $0 < \varepsilon_0 < 1$ such that*

$$\gamma_\varepsilon(\theta,t) \geq e^{-\frac{t-t_0}{|B|t_0}} \gamma_\varepsilon(\theta, t_0) \tag{2.0.13}$$

for all $0 < \varepsilon < \varepsilon_0$, $0 < t_0 \leq t \leq T < T_c$, and $\theta \in \mathbb{S}^{n-1}$.

Proof. Let $x_0 \in \Gamma_{f,\varepsilon}(t_0)$ with polar coordinates (r_0, θ), $r_0 > 0$, $\theta \in \mathbb{S}^{n-1}$. Then $r_0 = \gamma_\varepsilon(\theta, t_0)$ and $f(x_0, t_0) = \varepsilon$ and by the inequality (2.0.12), we have

$$f(e^{-\frac{t-t_0}{|B|t_0}} x_0, t) \leq e^{-\frac{C}{|B|T}(t-t_0)} f(x_0, t_0) \leq f(x_0, t_0) = \varepsilon = f(\gamma_\varepsilon(\theta, t_0)),$$

implying that

$$e^{-\frac{t-t_0}{|B|t_0}} \gamma_\varepsilon(\theta, t_0) \leq \gamma_\varepsilon(\theta, t).$$

□

Lemma 2.0.5. *There exist constants $A > 0$, $B < 0$, $C > 0$, and $D > 0$ such that*

$$-Cf(x,t) + Ax \cdot \nabla_x f(x,t) + (-D + Bt)f_t(x,t) \leq 0 \tag{2.0.14}$$

on $\{f(x,t) \leq 1, \ \ 0 \leq t \leq T\}$.

Proof. Let $t^* = \tau/2$ and $\mathcal{A}_{t^*} = \{f(x,t) \leq 1, t^* \leq t \leq T\}$. We want to show that for some negative constant B, and positive constants A, C, D, and for sufficiently small $\varepsilon > 0$,

$$f_\varepsilon(x,t) = \frac{1}{1+C\varepsilon} f((1+A\varepsilon)x, (1+B\varepsilon)t - D\varepsilon) \leq f(x,t)$$

on $\mathcal{A}_{t^*}$. We first choose $C = 1$ so that by Lemma 2.0.1, f_ε is a subsolution to the equation for the scalar curvature flow if $B + 1 - 4A < -|1 - A|$. Given $0 < A < 1$ to be determined later, take $B < 0$ such that $B < 5A - 2$. Then f_ε is a subsolution, especially in $\mathcal{A}_{t^*}$. Therefore, the comparison principle implies that it suffices to show that $f_\varepsilon \leq f$ on the parabolic boundary of $\mathcal{A}_{t^*}$, where $f(x,t) = 1, \ t^* \leq t \leq T$ or $f(x,t) \leq 1, \ t = t^*$. It is equivalent to show that $\frac{\partial f_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} = -f(x,t) + Ax \cdot \nabla_x f(x,t) + (-D + Bt)f_t(x,t) \leq 0$.

On $\{f(x,t) = 1, \ t^* \leq t \leq T\}$, we can take $A > 0$ sufficiently small that we have $-f(x,t) + Ax \cdot \nabla_x f(x,t) + (-D+Bt)f_t(x,t) = -1 + Ax \cdot \nabla_x f(x,t) \leq 0$ since $x \cdot \nabla_x f(x,t) \geq 0$ and $f(\cdot,t)$ is uniformly $C^{1,1}$ in $0 \leq t \leq T$, $f \leq 1$. Secondly, on the set $\{f(x,t) \leq 1, \ t = t^*\}$ and for the pressure-like function $g = \sqrt{2f}$, we see that $-f(x,t) + Ax \cdot \nabla_x f(x,t) + (-D + Bt)f_t(x,t) = g\{-\frac{1}{2}g(x,t^*) + Ax \cdot \nabla_x g(x,t^*) + (-D + Bt^*)g_t(x,t^*)\}$. Because $g(x,t^*) > 0$, $B < 0$, and $g_t(x,t^*) \geq 0$, we only need to show that $-g(x,t^*) + 2Ax \cdot \nabla_x g(x,t^*) - 2Dg_t(x,t^*) \leq 0$.

However, we assumed the initial non-degeneracy condition for f and hence for g so that for sufficiently small t^*

$$|\nabla_x g(x,t^*)| \geq c \ \text{ and } \ g_{\tau\tau} \geq c \ \text{ at the interface } \Gamma(t^*)$$

for some $c > 0$. For this reason there are some $\rho > 0$ and $c_0 > 0$ satisfying $g_t \geq c_0 > 0$, so that $-g(x,t^*) + 2Ax \cdot \nabla_x g(x,t^*) - 2Dg_t(x,t^*) \leq 0$ on the set $\{0 < g(x,t^*) < \sqrt{2}, \ d(x,\Gamma(t^*) < \rho\}$ if we take sufficiently small $A > 0$ since $D > 0$. The same estimate holds when $0 \leq t \leq t^*$. Hence $f_\varepsilon \leq f$ on $\{f(x,t) \leq 1, 0 \leq t \leq T\}$. □

Lemma 2.0.6. *There exists constants $A > 0$, $B < 0$ and $D > 0$ such that*

$$\gamma(\theta,t) \leq e^{-\frac{A(t-t_0)}{D+|B|T_c}} \gamma(\theta,t_0) \tag{2.0.15}$$

for $0 < t_0 \leq t \leq T < T_c$.

Proof. For time $0 < t_0 \leq t < T < T_c$, the inequality (2.0.14) implies that

$$\begin{aligned} 0 &\leq \frac{Cf(x,t)}{D+|B|t} - \frac{Ax}{D+|B|t} \cdot \nabla_x f(x,t) + f_t(x,t) \\ &\leq \frac{Cf(x,t)}{D+|B|t_0} - \frac{Ax}{D+|B|T_c} \cdot \nabla_x f(x,t) + f_t(x,t) \end{aligned}$$

by which we obtain

$$\frac{d}{dt}(e^{\frac{C(t-t_0)}{D+|B|t_0}} f(e^{-\frac{A(t-t_0)}{D+|B|T_c}}x,t)) \geq 0.$$

Hence we have

$$e^{\frac{C(t-t_0)}{D+|B|t_0}} f(e^{-\frac{A(t-t_0)}{D+|B|T_c}} x, t) \geq f(x, t_0) = 0 \tag{2.0.16}$$

which implies the conclusion. □

Lemma 2.0.7. *There exists constants $A > 0$, $B < 0$ and $D > 0$ such that for $0 < t_0 \leq t \leq T < T_c$ and $\theta \in \mathbb{S}^{n-1}$,*

$$\frac{-R}{|B|t_0} \leq \gamma_t(\theta, t) \leq -\frac{A\rho_0}{(D + |B|T_c)}. \tag{2.0.17}$$

Hence, the interface moves with a finite and non-degenerate speed for $0 < t_0 \leq t \leq T < T_c$.

Proof. Lemma 2.0.3 implies that for $0 < t_0 \leq t \leq t_1 \leq T < T_c$.

$$\frac{\gamma(\theta, t)(e^{-\frac{t_1-t}{|B|t}} - 1)}{t_1 - t} \leq \frac{\gamma(\theta, t_1) - \gamma(\theta, t)}{t_1 - t}.$$

And as $t_1 \to t^+$,

$$(\gamma)_t(\theta, t) \geq \gamma(\theta, t)\frac{-1}{|B|t} \geq \frac{-R}{|B|t_0}.$$

The other side is obtained in a similar way from Lemma 2.0.6. □

Theorem 2.0.8. *There exists a uniform constant $C > 1$ such that*

$$-C \leq (\gamma)_t(\theta, t) \leq -C^{-1} \tag{2.0.18}$$

for $0 \leq t \leq T < T_c$, $\theta \in \mathbb{S}^{n-1}$. Hence, the interface moves with a finite and non-degenerate speed.

Proof. By Theorem 1.6.2, there is a short time $T_0 > 0$ such that for time $0 \leq t \leq T_0$, a smooth solution for the flow exists and there is a uniform constant $C_0 > 1$ satisfying $-C_0 \leq (\gamma)_t(\theta, t)$ and hence (2.0.18) because of the right-hand side in (2.0.17). For time $T_0 \leq t \leq T < T_c$, we have Lemma 2.0.7. Hence we can pick a constant $C > 1$ such that (2.0.18) holds for all time $0 \leq t \leq T < T_c$. □

Lemma 2.0.9. *There exist uniform constants $B < 0$, $0 < \varepsilon_0 < 1$ such that*

$$(\gamma_\varepsilon)_t(\theta, t) \geq \frac{-R}{|B|t_0} \tag{2.0.19}$$

for $0 < \varepsilon < \varepsilon_0$, $0 < t_0 \leq t \leq T < T_c$, $\theta \in \mathbb{S}^{n-1}$.

Proof. The proof is the same as Lemma 2.0.7, but we use Lemma 2.0.4 instead of Lemma 2.0.3. □

Lemma 2.0.10. *There exist uniform constants $C > 1$, $0 < \varepsilon_0 < 1$ such that*

$$(\gamma_\varepsilon)_t(\theta, t) \geq -C \tag{2.0.20}$$

for $0 < \varepsilon < \varepsilon_0$, $0 \leq t \leq T < T_c$, $\theta \in \mathbb{S}^{n-1}$. Hence, the level set moves with a finite speed.

Proof. By Theorem 1.6.2, there is a short time $T_0 > 0$ such that for time $0 \leq t \leq T_0$, a smooth solution for the flow exists and there is a uniform constant $C_0 > 1$ satisfying $-C_0 \leq (\gamma_\varepsilon)_t(\theta, t)$. Hence (2.0.20) holds for time $0 \leq t \leq T < T_c$, because we have Lemma 2.0.9 for time $T_0 \leq t \leq T < T_c$. □

Chapter 3

Gradient estimates

3.1 Evolution of derivatives

Let us consider the following linear operator, $L[w]$, which will occur in the evolution of g_m and g_{mp}:

$$L[w] := \sum_{i,j=1}^{n} a_{ij} w_{ij} + \sum_{i=1}^{n} b_i w_i, \tag{3.1.1}$$

where the coefficients are defined by

$$\begin{aligned} a_{ii} &:= \frac{1}{I^{3/2}}\Big(\sum_{j \neq i} \big(g_j^2 + g g_{jj} + g^3 (g_k^2 g_{jj} - g_j g_k g_{kj})\big) - g^3 (g_i^2 g_{kk} - g_i g_k g_{ik})\Big), \\ a_{ij} &:= -\frac{1}{I^{3/2}}\big(g_i g_j + g g_{ij} + g^3 (g_k^2 g_{ij} + g_i g_j g_{kk} - g_i g_k g_{jk} - g_j g_k g_{ik})\big), \text{ for } i \neq j, \end{aligned} \tag{3.1.2}$$

$$\begin{aligned} b_i := \frac{1}{2I^{5/2}}\Big(& 4I(g_i g_{jj} - g_j g_{ij}) + 6g^5 g_i g_k g_l (g_{kl} g_{jj} - g_{jk} g_{jl}) \\ &+ g^2\big(- 6g_i (g_k^2 g_{jj} - g_k g_j g_{kj}) - 4Igg_k (g_{ik} g_{jj} - g_{jk} g_{ij}) - Igg_i (g_{kk} g_{jj} - g_{kj}^2)\big)\Big). \end{aligned} \tag{3.1.3}$$

Also, we define

$$c := \frac{1}{2I^{5/2}}\Big(I(g_{ii} g_{jj} - g_{ij}^2) - 6g^2 g_i g_k (g_{ik} g_{jj} - g_{jk} g_{ij}) - 6gg_i^2 (g_j^2 g_{kk} - g_k g_j g_{kj})\Big). \tag{3.1.4}$$

Lemma 3.1.1. *For $1 \leq m \leq n$, $(g_m)_t$ is given by*

$$(g_m)_t = L[g_m] + cg_m. \tag{3.1.5}$$

Proof. This equation (3.1.5) can be obtained by a direct calculation $(g_m)_t = (g_t)_m$. Let G_1 and G_2 be the two groups of terms of $g_t = G_1 + G_2$ (1.3.8):

$$\begin{aligned} G_1 &:= \frac{1}{2\sqrt{I}} \sum_{i,j=1}^{n} \big(g(g_{ii} g_{jj} - g_{ij}^2) + g_j^2 g_{ii} + g_i^2 g_{jj} - 2g_i g_j g_{ij}\big), \\ G_2 &:= -\frac{1}{I^{3/2}} g^2 \sum_{i,j,k=1}^{n} g_i g_k \big(g(g_{ik} g_{jj} - g_{jk} g_{ij}) + g_j^2 g_{ik} + g_i g_k g_{jj} - g_i g_j g_{jk} - g_j g_k g_{ij}\big). \end{aligned} \tag{3.1.6}$$

Differentiating with respect to any direction x_m, we get $(G_1)_m$ and $(G_2)_m$ as below:

$$\begin{aligned}(G_1)_m =&\frac{1}{\sqrt{I}}\Big(g(g_{ii}g_{jjm}-g_{ij}g_{ijm})+g_i^2g_{jjm}-g_ig_jg_{ijm}\Big)\\&+\frac{2}{\sqrt{I}}(g_jg_{ii}-g_ig_{ij})g_{jm}-\frac{1}{2I^{3/2}}\Big((g_{ii}g_{jj}-g_{ij}^2)g+g_j^2g_{ii}+g_i^2g_{jj}-2g_ig_jg_{ij}\Big)g^2g_kg_{km}\\&+\frac{1}{2\sqrt{I}}(g_{ii}g_{jj}-g_{ij}^2)g_m-\frac{1}{2I^{3/2}}\Big((g_{ii}g_{jj}-g_{ij}^2)g+g_j^2g_{ii}+g_i^2g_{jj}-2g_ig_jg_{ij}\Big)gg_k^2g_m,\end{aligned}\tag{3.1.7}$$

$$\begin{aligned}(G_2)_m =&-\frac{1}{I^{3/2}}g^2g_ig_k\Big(g(g_{ik}g_{jjm}+g_{jj}g_{ikm}-g_{jk}g_{ijm}-g_{ij}g_{jkm})+g_j^2g_{ikm}+g_ig_kg_{jjm}-g_ig_jg_{jkm}-g_jg_kg_{ijm}\Big)\\&-\frac{1}{I^{3/2}}g^2g_ig_k\Big(2g_jg_{ik}g_{jm}+g_ig_{jj}g_{km}+g_kg_{jj}g_{im}-g_ig_{jk}g_{jm}-g_kg_{ij}g_{jm}-g_jg_{jk}g_{im}-g_jg_{ij}g_{km}\Big)\\&+\frac{3}{I^{5/2}}g^4g_ig_k\Big(g(g_{ik}g_{jj}-g_{jk}g_{ij})+g_j^2g_{ik}+g_ig_kg_{jj}-g_ig_jg_{jk}-g_jg_kg_{ij}\Big)g_lg_{lm}\\&-\frac{1}{I^{3/2}}g^2\Big(g(g_{ik}g_{jj}-g_{jk}g_{ij})+g_j^2g_{ik}+g_ig_kg_{jj}-g_ig_jg_{jk}-g_jg_kg_{ij}\Big)(g_ig_{km}+g_kg_{im})\\&+\frac{1}{I^{3/2}}gg_ig_k\Big(g_j^2g_{ik}+g_ig_kg_{jj}-g_ig_jg_{jk}-g_jg_kg_{ij}\Big)g_m\\&-\frac{1}{I^{5/2}}gg_ig_k\Big(3g(g_{ik}g_{jj}-g_{jk}g_{ij})+3g_j^2g_{ik}+3g_ig_kg_{jj}-3g_ig_jg_{jk}-3g_jg_kg_{ij}\Big)g_m.\end{aligned}\tag{3.1.8}$$

Adding $(G_1)_m$ and $(G_2)_m$ and arranging the terms, we have the equation (3.1.5). □

3.2 Gradient estimates

Now, we get estimates of the gradient of $g=\sqrt{2f}$ from above and below. If $g\geq\varepsilon>0$, then $|Dg(x,t)|$ is bounded from above and below because $f(x,t)$ is smooth and strictly convex on $\{\frac{\varepsilon}{2}<g<2\}$ by Kim and Lee [11]. Hence it is sufficient to consider $0\leq g<\varepsilon$ for some $0<\varepsilon\leq\varepsilon_0$ where $\varepsilon_0<1$ is the constant in Lemma 2.0.4, 2.0.9, 2.0.10. We also show that the level set moves with a non-degenerate speed.

Throughout the section, we assume that g is a solution of (1.3.8) and it is smooth up to the interface in its support for $0\leq t\leq T<T_c$. Let $P=(x,t)\in\Omega(t)$. Let $\mathbf{P}$ be the position vector of P with respect to the origin 0. Because $B(0,\rho_0)$ is contained in the flat side $\Sigma_0(t)$ by the assumption (A5) and $\mathbf{P}\cdot\nu(P)$ is the distance of the origin from the tangent hyperplane of $\Gamma_P(t)$ at P, we get the following.

Lemma 3.2.1. *Let us assume the conditions in the subsection 1.4. Then we have*

$$\boldsymbol{P}\cdot\nu(P)\geq\rho_0,\ \textit{for any}\ P=(x,t)\in\Omega(t),\ 0\leq t\leq T.\tag{3.2.1}$$

Then we obtain the following result, as well.

Lemma 3.2.2. *Let us assume the conditions in the subsection 1.4. Let $P_0=(x_0,t_0)$ be an arbitrary point on the interface $\Gamma(t_0)$ with $0<t_0\leq T$. We define its position vector $\mathbf{P_0}$ and normalize it with $n_0=\frac{\mathbf{P_0}}{|\mathbf{P_0}|}$. Then there exists positive constants η and γ such that*

$$n_0\cdot\nu(P)\geq\gamma,\ \textit{for any}\ P=(x,t)\in\Omega(t)\ \textit{with}\ \ |P-P_0|\leq\eta,\ 0\leq t\leq t_0.\tag{3.2.2}$$

Proof. Note that $|\mathbf{P}| \leq C$ for some constant $C > 0$ which depends on initial data. With Lemma 3.2.1, it follows that

$$\frac{1}{|\mathbf{P}|}\mathbf{P} \cdot \nu(P) \geq C^{-1}\rho_0. \tag{3.2.3}$$

Setting $\gamma = C^{-1}\rho_0/2$ and choosing η sufficiently small depending on ρ_0 and γ, we get (3.2.2). □

Lemma 3.2.3. *Let us assume the conditions in the subsection 1.4. Then there exists a constant* $C > 0$ *such that*

$$|\nabla g| \leq C, \text{ on } 0 \leq g(\cdot, t) \leq 1,\ 0 \leq t \leq T. \tag{3.2.4}$$

Proof. We may assume that $0 \leq g < \varepsilon_0$, because of the existence of a stricly convex smooth solution from [11]. Because the equation (3.1.5) for the first derivatives of g is parabolic, we can use the maximum principle to get the uniform upper bound for $|\nabla g|$. First, consider the whole scalar curvature flow with a flat side. The flat side will keep shrinking to a point in a finite time T_c and the strictly convex side will become smooth and keep being strictly convex, by [11].

Let us approximate f by a decreasing sequence of solutions f_ε, $\varepsilon \to 0^+$ of the equation (1.3.7) and set $g_\varepsilon = \sqrt{2f_\varepsilon}$. We can choose f_ε's such that $|\nabla g_\varepsilon(\cdot, 0)| \leq C$ on the set $\{x \in \mathbb{R}^n : 0 \leq g_\varepsilon(x, 0) \leq 1\}$ at time $t = 0$ by the assumption (A4) and $|\nabla g_\varepsilon(\cdot, t)| \leq C$ on the level set $\Gamma_1(t) = \{x \in \mathbb{R}^n : g_\varepsilon(x, t) = 1\}$ for time $0 \leq t \leq T$ by (A4), the estimate (1.5.1), and Theorem 1.6.2.

Therefore, it is enough to show that the uniform estimate (3.2.4) for g_ε holds in the interior. To simplify the notation, we denote g_ε by g from now on. Let us set $X := \frac{1}{2}|\nabla g|^2 = \frac{1}{2}\sum_i g_i^2$. Then $X = \frac{1}{2}g_\nu^2$ for the normal vector ν to the level set of g. Suppose that at each time $0 \leq t < T$, X attains an interior maximum at $P_0 = (x_0, t)$. Rotating the coordinates, we can make $g_1 = g_\nu > 0$, $g_i = 0$ for $2 \leq i \leq n$, $g_{ij} = 0$ for $2 \leq i, j \leq n$ with $i \neq j$ at P_0. As

$$X_1 = g_1 g_{11} + \sum_{i\geq 2} g_i g_{i1} = g_1 g_{11} = 0,\ X_i = g_1 g_{1i} + \sum_{j \geq 2} g_j g_{ji} = g_1 g_{1i} = 0,\ i \geq 2, \tag{3.2.5}$$

we have $g_{11} = g_{1i} = 0$ for $2 \leq i \leq n$ at P_0. Note that $g_{ii} \geq 0$, $2 \leq i \leq n$ by convexity of level sets of g. Let us look at the second-order derivatives of X. At P_0, we have

$$\begin{aligned} X_{11} &= g_1 g_{111} + g_{11}^2 + \sum_{i\geq 2}(g_i g_{i11} + g_{i1}^2) = g_1 g_{111} \leq 0, \\ X_{ii} &= g_1 g_{1ii} + g_{1i}^2 + \sum_{j\geq 2}(g_j g_{jii} + g_{ji}^2) = g_1 g_{1ii} + \sum_{j\geq 2} g_{ji}^2 \leq 0,\ 2 \leq i \leq n, \\ X_{ij} &= g_1 g_{1ij} + g_{1i} g_{1j} + \sum_{k\geq 2}(g_k g_{kij} + g_{ki} g_{kj}) = g_1 g_{1ij} + \sum_{k \geq 2} g_{ki} g_{kj} \leq 0,\ 1 \leq i, j \leq n, \end{aligned} \tag{3.2.6}$$

so that $g_1 g_{111} \leq 0$ and $g_1 g_{1ii} \leq 0$.

X is invariant under coordinate rotations. In the local coordinates at P_0 where $g_1 = g_\nu$,

$g_i = 0$ for $i \neq 1$ and $g_{ij} = 0$ for $i \neq j$, the evolution of X at P_0 is given by

$$X_t = \Big(\frac{gg_1H_{\Gamma_\varepsilon}}{I^{3/2}} + \frac{gg_1H_{\Gamma_\varepsilon}}{\sqrt{I}} + \frac{g_1^2}{I^{3/2}}\Big)X_{11} + \sum_{i=2}^{n}\Big(\frac{gg_1H_{\Gamma_\varepsilon}}{\sqrt{I}} + \frac{g_1^2}{I^{3/2}}\Big)X_{ii} - \sum_{i,j=2}^{n}\frac{gg_{ij}}{\sqrt{I}}X_{ij}$$
$$+ \frac{-3gg_1^6}{I^{5/2}}g_1H_{\Gamma_\varepsilon} + \frac{1}{2I^{3/2}}g_1^2(g_1^2H_{\Gamma_\varepsilon}^2 - \sum_{i,j=2}^{n}g_{ij}^2) - \sum_{i=2}^{n}\Big(\frac{gg_1H_{\Gamma_\varepsilon}}{\sqrt{I}} + \frac{g_1^2}{I^{3/2}}\Big)\sum_{j=2}^{n}g_{ji}^2 + \sum_{i,j=2}^{n}\frac{gg_{ij}}{\sqrt{I}}\sum_{k=2}^{n}g_{ki}g_{kj} \tag{3.2.7}$$

where $H_{\Gamma_\varepsilon} = \frac{1}{g_1}\sum_{i=2}^{n} g_{ii}$ is the mean curvature of the level set $\Gamma_\varepsilon(t)$ at P_0.

Then, since $X_i = 0$ and $X_{ii} \leq 0$ for all $1 \leq i \leq n$ at P_0,

$$X_t = a_{11}X_{11} + \sum_{i=2}^{n}a_{ii}X_{ii} + \frac{g_1^2}{2I^{3/2}}(g_1^2H_{\Gamma_\varepsilon}^2 - \sum_{i=2}^{n}g_{ii}^2) + \frac{-3gg_1^6}{I^{5/2}}g_1H_{\Gamma_\varepsilon} - \sum_{i=2}^{n}\Big(\frac{gg_1H_{\Gamma_\varepsilon}}{\sqrt{I}} + \frac{g_1^2}{I^{3/2}}\Big)g_{ii}^2 + \sum_{i=2}^{n}\frac{gg_{ii}^3}{\sqrt{I}}$$

where $a_{11} = \frac{gg_1H_{\Gamma_\varepsilon}}{I^{3/2}} + \frac{gg_1H_{\Gamma_\varepsilon}}{\sqrt{I}} + \frac{g_1^2}{I^{3/2}} \geq 0$ and $a_{ii} = \frac{g}{\sqrt{I}}\sum_{j\neq 1,i} g_{jj} + \frac{g_1^2}{I^{3/2}} \geq 0$ for $2 \leq i \leq n$, so that

$$X_t \leq \frac{g_1^4}{2I^{3/2}}H_{\Gamma_\varepsilon}^2. \tag{3.2.8}$$

On the level set $\Gamma_\varepsilon(t)$,

$$\begin{aligned} &g_r\dot{\gamma}(t) + g_t = 0, \quad g_1 = g_rD_{x_1}r = g_r\frac{x_1}{\gamma}, \\ &\frac{g_t}{g_1} = -\frac{\gamma\dot{\gamma}(t_0)}{x_1} \leq \frac{C\gamma}{x_1} \leq \frac{CR}{\rho_0} = C_2 \end{aligned} \tag{3.2.9}$$

by the assumptions (A3), (A5), Lemma 2.0.10 and Lemma 3.2.1 that $x_1 = \mathbf{P}\cdot\nu \geq \rho_0$. Hence at P_0

$$\frac{g_t}{g_1} = \frac{1}{2g_1\sqrt{I}}g\sum_{i,j=2}^{n}(g_{ii}g_{jj} - g_{ij}^2) + \frac{1}{g_1I^{3/2}}\sum_{i=2}^{n}\Big(g(g_{ii}g_{11} - g_{i1}^2) + g_1^2g_{ii}\Big) \leq C_2. \tag{3.2.10}$$

At P_0, $g_{11} = g_{i1} = 0$ so we have

$$\frac{g_t}{g_1} = \frac{g}{2g_1\sqrt{I}}\sum_{i,j=2}^{n}(g_{ii}g_{jj} - g_{ii}^2) + \frac{g_1^2}{I^{3/2}}H_{\Gamma_\varepsilon} \leq C_2, \tag{3.2.11}$$

and for some uniform constant $C > 0$ depending on C_2, R and M from (1.5.1) ,

$$\begin{aligned} X_t \leq &\frac{g_1^4}{2I^{3/2}}H_{\Gamma_\varepsilon}^2 = \frac{I^{3/2}}{2}\Big(\frac{g_1^2}{I^{3/2}}H_{\Gamma_\varepsilon}\Big)^2 \leq \frac{1}{2}C_2^2I^{3/2} = \frac{1}{2}C_2^2(1 + |\nabla f|^2)^{3/2} \\ \leq &\frac{1}{2}C_2^2(1 + M^2d(x,\Gamma(t))^2)^{3/2} \leq C. \end{aligned} \tag{3.2.12}$$

Hence, $X(t)$ is bounded above by the solution $Y(t)$ to the ODE $\frac{d}{dt}Y = C,\ Y(0) = X(0)$ so that

$$X(t) \le Y(t) = Ct \le CT \le CT_c \tag{3.2.13}$$

for time $0 \le t \le T$ by the maximum principle. □

Lemma 3.2.4. *Let us assume the conditions in the subsection 1.4. Then there exists a constant $C > 0$ such that*

$$|\nabla g| \ge C \text{ on } g(\cdot, t) > 0,\ 0 \le t \le T. \tag{3.2.14}$$

Proof. At the initial time $t = 0$, $|\nabla g|$ is bounded below near the free boundary Γ by the assumption (A4). Consequently, for a sufficiently small $0 < \varepsilon \le \varepsilon_0$, it holds that

$$|\nabla g| \ge \frac{\lambda}{2} \text{ on } 0 \le g(\cdot, 0) \le \varepsilon \tag{3.2.15}$$

for the constant $\lambda > 0$ in the assumption (A4) (1.4.1). We may assume that $0 \le g < \varepsilon$, because of the existence of a stricly convex smooth solution from [11].

Let us define a quantity $X := x \cdot \nabla g - kx \cdot \nabla f$, where k is a positive constant that will be determined later. We will prove by the maximum principle that if $X \ge c_0 > 0$ at time $t = 0$ then

$$X \ge c > 0 \text{ on } g(\cdot, t) > 0,\ 0 \le t \le T, \tag{3.2.16}$$

and this will imply the lemma.

Suppose that at each time $0 \le t < T$, X attains an interior minimum at $P_0 = (x_0, t)$ so that $X(x_0, t) = \inf\{X(x, t);\ x \in \Sigma,\ 0 < g(x, t) \le 1\}$. At P_0 on Γ_ε, we choose local coordinates such that $x_1 > 0$ and $x_i = 0$ for $i = 2, \dots, n$ and $g_{ij} = 0$ for all $i \ne j$.

Then the inequality (3.2.10) at P_0 becomes

$$\frac{g_t}{g_1} = \frac{1}{2g_1\sqrt{I}} g \sum_{i,j=2}^{n} g_{ii} g_{jj} + \frac{1}{g_1 I^{3/2}} \sum_{i=2}^{n} (g g_{ii} g_{11} + g_1^2 g_{ii}) = \frac{1}{2g_1\sqrt{I}} g \overline{R}_{g,2} + \frac{1}{I^{3/2}} g_1 \sum_{i=2}^{n} g_{ii} \le C_2, \tag{3.2.17}$$

where

$$\overline{R}_{g,2} = \sum_{i,j=2}^{n} (g_{ii} g_{jj} - g_{ij}^2) + \frac{2}{I} \sum_{i=2}^{n} (g_{ii} g_{11} - g_{1i}^2). \tag{3.2.18}$$

At P_0, $x \cdot \nabla f = gx \cdot \nabla g = gx_1 g_1 + gx_i g_i = gx_1 g_1$ in our local coordinates where $x_i = 0$ for $i = 2 \dots n$, so we have $X = x_1 g_1 - kgx_1 g_1$ and the first-order derivatives of X are

$$\begin{aligned} X_1 &= g_1 + x_1 g_{11} - k(g_1^2 x_1 + gg_1 + gx_1 g_{11}), \\ X_i &= g_i + x_1 g_{1i} - k(g_i x_1 g_1 + gg_i + gx_1 g_{1i}) = g_i(1 - kx_1 g_1 - kg) \text{ for } i = 2, \dots, n, \end{aligned} \tag{3.2.19}$$

with $X_1 = 0$, $X_i = 0$ for $i \ne 1$, hence $g_i = 0$ for $i \ne 1$. And its second-order derivatives at P_0 are

$$\begin{aligned} X_{11} &= x_1 g_{111} + 2g_{11} - k(3x_1 g_1 g_{11} + 2g_1^2 + 2gg_{11} + gx_1 g_{111}), \\ X_{1i} &= x_1 g_{11i} + 2g_{1i} - k(2g_{1i} x_1 g_1 + 2gg_{1i} + gx_1 g_{11i}),\ i = 2, \dots, n, \\ X_{ij} &= x_1 g_{1ij} + 2g_{ij} - k(g_{ij} x_1 g_1 + gg_{ij} + gx_1 g_{1ij}),\ i, j = 2, \dots, n. \end{aligned} \tag{3.2.20}$$

with $X_{11} \ge 0$, $X_{ii} \ge 0$ for $i = 2, \dots, n$.

Then the evolution equation of X at the point P_0 can be written as

$$\begin{aligned}X_t =& \Big(\frac{g\sum_{j=2}^n g_{jj}}{\sqrt{I}} + \frac{g_1^2 + gg_{11}}{I^{3/2}}\Big)\sum_{i=2}^n (X_{ii} - 2g_{ii} + kg_{ii}x_1g_1 + kgg_{ii}) + \frac{x_1g_1}{2\sqrt{I}}\overline{R}_{g,2} - \frac{kgx_1g_1}{2\sqrt{I}}\overline{R}_{g,2}\\ &+ \frac{g\sum_{i=2}^n g_{ii}}{I^{3/2}}(X_{11} - 2g_{11} + 3kx_1g_1g_{11} + 2kg_1^2 + 2kgg_{11})\\ &+ \frac{2g_1\sum_{i=2}^n g_{ii}}{I^{3/2}}(X_1 - g_1 + kg_1^2x_1 + kgg_1) - \left(\frac{1}{2\sqrt{I}}g\overline{R}_{g,2} + \frac{1}{I^{3/2}}g_1^2\sum_{i=2}^n g_{ii}\right)kx_1g_1\\ &- \left(\frac{1}{2I^{3/2}}g\overline{R}_{g,2} + \frac{1}{I^{5/2}}g_1^2\sum_{i=2}^n g_{ii}\right)x_1gg_1(1-kg)(g_1^2 + gg_{11}) - \frac{2x_1gg_1(1-kg)(g_1^2+gg_{11})^2}{I^{5/2}}\sum_{i=2}^n g_{ii}\end{aligned} \tag{3.2.21}$$

which implies that

$$\begin{aligned}X_t \geq& \frac{kx_1g_1}{I^{3/2}}(g_1^2 + 3gg_{11})\sum_{i=2}^n g_{ii} + \frac{kg}{I^{3/2}}(3g_1^2 + gg_{11})\sum_{i=2}^n g_{ii}\\ &- 3C_2g_1 - C_2kx_1g_1^2 - \frac{3C_2}{I}x_1gg_1^2(g_1^2 + gg_{11}) + \frac{x_1g_1}{2\sqrt{I}}\overline{R}_{g,2} + \frac{2kx_1g^2g_1}{I^{5/2}}(g_1^2 + gg_{11})^2\sum_{i=2}^n g_{ii}\end{aligned} \tag{3.2.22}$$

On the other hand, we have at P_0

$$0 = X_1 = x_1(1-kg)g_{11} + g_1 - k(g_1^2x_1 + gg_1) \tag{3.2.23}$$

so by letting $g \leq \frac{1}{2k}$, Lemma 3.2.1 and 3.2.3 imply that

$$g_{11} = \frac{-g_1 + k(g_1^2x_1 + gg_1)}{x_1(1-kg)} \leq \frac{2k(C^2R + C)}{\rho_0} \tag{3.2.24}$$

and that

$$\begin{aligned}X_t &\geq -C_2\left(kC + \frac{3}{\rho_0} + 3C^3 + 3C\frac{2k(C^2R+C)}{\rho_0}\right)X - \left(kC + \frac{3}{\rho_0} + 3C^3 + 3C\frac{2k(C^2R+C)}{\rho_0}\right)k\rho_0C_2C\\ &= -\alpha X - k\beta.\end{aligned} \tag{3.2.25}$$

Hence, by Grönwall's inequality, we obtain

$$\min_{x\in M}X(x,t) \geq e^{-\alpha t}(\min_{x\in M}X(x,0) - k\beta t) \geq \frac{1}{2}e^{-\alpha T_c}\min_{x\in M}X(x,0) =: C', \tag{3.2.26}$$

if we take $k \leq \frac{\min_{x\in M}X(x,0)}{2\beta T}$. Then $Y = x_1g_1 \geq X \geq C'$ and $|\nabla g| = g_1 \geq \frac{C'}{R}$.

Finally, consider the case when the minimum of X is attained at a point $P_0(t)$ on the free boundary $\Gamma(t)$, where $g = 0$. Since X is rotationally invariant, we can pick the coordinate system where $g_1 = g_\nu > 0$, $g_i = 0$ for $i \neq 1$. Then the first-order derivatives of X at P_0 are

$$\begin{aligned}X_1 &= g_1 + x_1g_{11} - k(g_1^2x_1 + gg_1 + gx_1g_{11}) = g_1 + x_1g_{11} - kg_1^2x_1 \geq 0,\\ X_i &= g_i + x_1g_{1i} - k(g_ix_1g_1 + gg_i + gx_1g_{1i}) = x_1g_{1i} = 0 \text{ for } i = 2,\dots,n,\end{aligned} \tag{3.2.27}$$

so that $g_{1i} = 0$ for $i = 2, \dots, n$, at P_0. And its second-order derivatives at P_0 are

$$\begin{aligned}
X_{11} &= x_1 g_{111} + 2g_{11} - k(3x_1 g_1 g_{11} + 2g_1^2 + 2gg_{11} + gx_1 g_{111}),\\
X_{1i} &= x_1 g_{11i} + 2g_{1i} - k(2g_{1i}x_1 g_1 + 2gg_{1i} + gx_1 g_{11i}),\ i = 2, \dots, n,\\
X_{ij} &= x_1 g_{1ij} + 2g_{ij} - k(g_{ij}x_1 g_1 + gg_{ij} + gx_1 g_{1ij}),\ i, j = 2, \dots, n,\\
X_{ii} &= x_1 g_{1ii} + 2g_{ii} - k(g_{ii}x_1 g_1 + gg_{ii} + gx_1 g_{1ii}),\ i = 2, \dots, n.
\end{aligned} \tag{3.2.28}$$

and it holds at P_0 that

$$X_{ii} = x_1 g_{1ii} + 2g_{ii} - kg_{ii}x_1 g_1 \geq 0,\ i = 2, \dots, n. \tag{3.2.29}$$

As $X_1 \geq 0$ at P_0, it holds that

$$x_1 g_{11} \geq -g_1 + kx_1 g_1^2. \tag{3.2.30}$$

At P_0 where $g = 0$, we have

$$\begin{aligned}
X_t =& \frac{g_1^2}{I^{3/2}} \sum_{i=2}^{n} (X_{ii} - 2g_{ii} + kg_{ii}x_1 g_1) + \frac{x_1 g_1}{2\sqrt{I}} \overline{R}_{g,2}\\
&+ \frac{2g_1 \sum_{i=2}^{n} g_{ii}}{I^{3/2}} (X_1 - g_1 + kg_1^2 x_1) - \frac{1}{I^{3/2}} g_1^2 \sum_{i=2} g_{ii} kx_1 g_1.
\end{aligned} \tag{3.2.31}$$

However, the evolution of $X(P_0(t), t)$ on the free boundary is affected by the motion of the free boundary too. For any point $\gamma(t)$ on the $(n-1)$-dimensional free-boundary hypersurface $\Gamma(t)$ of the flow, we have

$$\frac{d}{dt} X(\gamma(t), t) = X_t + \nabla_x X \cdot \dot{\gamma}(t). \tag{3.2.32}$$

Since $g(\gamma(t), t) = 0$ so that $g_t = -\nabla_x g \cdot \dot{\gamma}(t) = -g_1 \dot{\gamma}_1(t)$ at $P_0(t) = \gamma(t)$, the term $\nabla_x X \cdot \dot{\gamma}(t)$ is given by

$$\nabla_x X \cdot \dot{\gamma}(t) = X_1 \dot{\gamma}_1(t) = -\frac{g_t}{g_1} X_1 \tag{3.2.33}$$

which implies that for $P_0(t) = \gamma(t) \in \Gamma(t)$

$$\begin{aligned}
\frac{d}{dt} X(\gamma(t), t) =& \frac{g_1^2}{I^{3/2}} \sum_{i=2}^{n} X_{ii} + \frac{2g_1 \sum_{i=2}^{n} g_{ii}}{I^{3/2}} X_1 + \frac{2kx_1 g_1^3}{I^{3/2}} \sum_{i=2}^{n} g_{ii} - \frac{4g_1^2}{I^{3/2}} \sum_{i=2}^{n} g_{ii} + \frac{x_1 g_1}{2\sqrt{I}} \overline{R}_{g,2} - \frac{g_t}{g_1} X_1\\
\geq& \frac{3kx_1 g_1^3}{I^{3/2}} \sum_{i=2}^{n} g_{ii} - \frac{5g_1^2}{I^{3/2}} \sum_{i=2}^{n} g_{ii}
\end{aligned} \tag{3.2.34}$$

The inequality (3.2.10) becomes at $P_0(t)$ where $g = 0$

$$0 \leq \frac{g_t}{g_1} = \frac{g_1}{I^{3/2}} \sum_{i=2}^{n} g_{ii} \leq C_2. \tag{3.2.35}$$

Thus at $P_0(t)$,

$$\begin{aligned}\frac{d}{dt}X(\gamma(t),t) \geq& \frac{3kx_1g_1g_1^2}{I^{3/2}}\sum_{i=2}^{n} g_{ii} - \frac{5g_1^2}{I^{3/2}}\sum_{i=2}^{n} g_{ii} = \frac{g_1^2}{I^{3/2}}\sum_{i=2}^{n} g_{ii}(3kx_1g_1 - 5)\\ \geq& -\frac{5g_1^2}{I^{3/2}}\sum_{i=2}^{n} g_{ii} \geq -5C_2g_1 \geq -\frac{5C_2X}{\rho_0},\end{aligned} \tag{3.2.36}$$

and $X(P_0(t),t)$ is bounded below by the solution $Y(t)$ to the ODE $\frac{d}{dt}Y = -\frac{5C_2Y}{\rho_0}$, $Y(0) = X(0)$ so that

$$X(P_0(t),t) \geq Y(t) = Y(0)e^{-\frac{5C_2t}{\rho_0}} \geq Y(0)e^{-\frac{5C_2T_c}{\rho_0}}. \tag{3.2.37}$$

for time $0 \leq t \leq T$. Hence, we get the uniform bound (3.2.14) by the maximum principle. □

Lemma 3.2.5. *There exists constants $A > 0$, $B < 0$, $D > 0$ and $0 < \varepsilon_0 < 1$ such that*

$$(\gamma_\varepsilon)_t(\theta,t) \leq -\frac{A\rho_0}{2(D + |B|T_c)} \tag{3.2.38}$$

for $0 < \varepsilon < \varepsilon_0$, $0 < t_0 \leq t \leq T$, $\theta \in \mathbb{S}^{n-1}$.

Proof. Fix $\theta \in \mathbb{S}^{n-1}$. Let $x_0 \in \Gamma_{f,\frac{1}{2}\varepsilon^2}(t) = \Gamma_\varepsilon(t)$ be the point with polar coordinates $(\gamma_\varepsilon(\theta,t),\theta)$. Then $f(x_0,t) = \frac{1}{2}\varepsilon^2$ and by the inequality (2.0.16), for $0 < t_0 \leq t \leq t_1 \leq T < T_c$,

$$e^{\frac{C(t_1-t)}{D+|B|t}} f(e^{-\frac{A(t_1-t)}{D+|B|T_c}}x_0, t_1) \geq f(x_0,t) = \frac{1}{2}\varepsilon^2,\ g(e^{-\frac{A(t_1-t)}{D+|B|T_c}}x_0,t_1) \geq \varepsilon e^{-\frac{C(t_1-t)}{2(D+|B|t_0)}}. \tag{3.2.39}$$

Let us use polar coordinates for simplicity. For any small $\delta > 0$, we have a Taylor expansion of g in the radius near the point $P(\theta,t_1) = (e^{-\frac{A(t_1-t)}{D+|B|T_c}}\gamma_\varepsilon(\theta,t),\theta,t_1)$ which is the expression of the point $(e^{-\frac{A(t_1-t)}{D+|B|T}}x_0,t_1)$ in polar coordinates:

$$g(e^{-\frac{A(t_1-t)}{D+|B|T_c}}\gamma_\varepsilon(\theta,t) + \delta,\theta,t_1) = g(P(\theta,t_1)) + \delta g_r(P(\theta,t_1)) + \delta O(\delta). \tag{3.2.40}$$

By Lemma (3.2.4), we have $g_r(P(\theta,t_1)) \geq 2c > 0$ for some $c > 0$, so

$$\begin{aligned}g(e^{-\frac{A(t_1-t)}{D+|B|T_c}}\gamma_\varepsilon(\theta,t) + \delta,\theta,t_1) &\geq \varepsilon e^{-\frac{C(t_1-t)}{2(D+|B|t_0)}} + 2c\delta g_r(P(\theta,t_1)) + \delta O(\delta) \geq \varepsilon, \text{ and}\\ \gamma_\varepsilon(\theta,t_1) &\leq e^{-\frac{A(t_1-t)}{D+|B|T_c}}\gamma_\varepsilon(\theta,t) + \delta.\end{aligned} \tag{3.2.41}$$

If we set $\delta = \frac{\varepsilon(1-e^{-\frac{C(t_1-t)}{2(D+|B|t_0)}})}{c}$, then δ has a Taylor expansion in time near t

$$\delta = \frac{\varepsilon(1 - e^{-\frac{C(t_1-t)}{2(D+|B|t_0)}})}{c} = \frac{\varepsilon\frac{C}{2(D+|B|t_0)}}{c}(t_1 - t) + (t_1 - t)O(t_1 - t). \tag{3.2.42}$$

As $t_1 \to t^+$, we have

$$\frac{\gamma_\varepsilon(\theta, t_1) - \gamma_\varepsilon(\theta, t)}{t_1 - t} \le \frac{e^{-\frac{A(t_1-t)}{D+|B|T_c}}}{t_1 - t}\gamma_\varepsilon(\theta, t) + \frac{\varepsilon^{\frac{C}{2(D+|B|t_0)}}}{c} + O(t_1 - t), \tag{3.2.43}$$

and for sufficiently small $\varepsilon > 0$, we have $(\gamma_\varepsilon)_t(\theta, t) \le -\frac{A}{D+|B|T_c}\gamma_\varepsilon(\theta, t) + \frac{\varepsilon^{\frac{C}{(D+|B|t_0)}}}{c} \le -\frac{A\rho_0}{2(D+|B|T_c)}$. □

Theorem 3.2.6. *There exist uniform constants $C > 1$ and $0 < \varepsilon_0 < 1$ such that*

$$-C \le (\gamma_\varepsilon)_t(\theta, t) \le -C^{-1} \tag{3.2.44}$$

for $0 < \varepsilon < \varepsilon_0,\ 0 \le t \le T,\ \theta \in \mathbb{S}^{n-1}$. Hence, the level set moves with a finite and non-degenerate speed.

Proof. The proof is the same as Theorem 2.0.8 but we use Lemma 2.0.9 and 3.2.5 instead of Lemma 2.0.7. Like the proof of Theorem 2.0.8, we use the short time existence, Theorem 1.6.2. □

Corollary 3.2.7. *Under the assumptions in the subsection 1.4, there exists a constant $c > 0$ for which*

$$c \le g_t \le c^{-1} \textit{ for time } 0 \le t \le T. \tag{3.2.45}$$

Proof. It sufficies to prove (3.2.45) on $0 \le g \le \varepsilon_0$, because of the existence of a stricly convex smooth solution from [11]. On $\Gamma(t)$ and $\Gamma_\varepsilon(t)$, we have

$$g_r\dot{\gamma}(\theta, t) + g_t = 0 \tag{3.2.46}$$

so that

$$g_t = -g_r\dot{\gamma}(\theta, t) \tag{3.2.47}$$

in which $\dot{\gamma}(\theta, t)$ is bounded by two negative constants by Theorem 2.0.8 and 3.2.6. On the other hand, g_r is bounded because

$$g_r = \nabla_x g \cdot \partial_r x = \sum_{i=1}^{n} \frac{rg_i}{x_i} = \frac{rg_\nu}{x_\nu} \tag{3.2.48}$$

where ν is the unit normal vector to the level set, g_ν is bounded by Lemma 3.2.3 and 3.2.4, and $\rho_0 \le x_\nu = \mathbf{P} \cdot \nu(P) \le R$ by Lemma 3.2.1 and our assumption in the subsection 1.4. Hence we get the result. □

Corollary 3.2.8. *Under the assumptions in the subsection 1.4, there exists a constant $c > 0$ such that*

$$c \le \frac{\sigma_2}{g} \le c^{-1},\ c \le \frac{1}{g}\sum_{i,j=1}^{n}(f_{ii}f_{jj} - f_{ij}^2) \le c^{-1} \textit{ for time } 0 \le t \le T. \tag{3.2.49}$$

Proof. Recall that, in the coordinates where $g_i = 0$ for $i \neq 1$ so that $f_i = 0$ for $i \neq 1$,

$$f_t = \frac{1}{\sqrt{1+|\nabla_x f|^2}}\left(1 - \frac{f_1^2}{1+|\nabla_x f|^2}\right)\sum_{i=2}^{n}(f_{ii}f_{11} - f_{i1}^2) + \frac{1}{2\sqrt{1+|\nabla_x f|^2}}\sum_{i,j=2}^{n}(f_{ii}f_{jj} - f_{ij}^2). \quad (3.2.50)$$

By convexity of $f(\cdot, t)$ from [11], we have

$$\frac{1}{2(1+|\nabla_x f|^2)^{3/2}}\sum_{i,j=1}^{n}(f_{ii}f_{jj} - f_{ij}^2) \leq f_t = \sigma_2\sqrt{1+|\nabla_x f|^2} \leq \frac{1}{2\sqrt{1+|\nabla_x f|^2}}\sum_{i,j=1}^{n}(f_{ii}f_{jj} - f_{ij}^2). \quad (3.2.51)$$

Since $f_t = gg_t$, we get (3.2.49) by (1.5.1) and Corollary 3.2.7. □

Chapter 4

Second-order derivative estimates

The purpose of this section is to obtain the optimal bound of the second-order tangential derivatives of g which gives the optimal decay rates of the second-order derivatives of f. Because of the existence of a stricly convex smooth solution from [11], it is sufficient to consider values of $0 \leq g < \varepsilon$, for some sufficiently small $0 < \varepsilon \leq \varepsilon_0$ where $0 < \varepsilon_0 < 1$ is the constant in Lemma 2.0.4, 2.0.9, 2.0.10.

Throughout the section, we assume that g is a solution of (1.3.8) and it is smooth up to the interface in its support for time $0 \leq t \leq T < T_c$. We get estimates on second-order derivatives of g first, under the assumptions that we have given in the subsection 1.4. We also find a lower bound of $R_{g,2}$ in Lemma 4.2.5 which is a generalized version of Kim-Lee-Rhee's curvature lower bound, [12]. It is also similar to the Aronson-Bénilan inequality $\Delta u \geq -\frac{C}{t}$ for the porous medium equation [1].

4.1 Evolution of second-order derivatives

Lemma 4.1.1. *For $1 \leq m, p \leq n$, $(g_{mp})_t$ is given by*

$$(g_{mp})_t = \sum_{i,j=1}^{n} a_{ij} g_{mpij} + \sum_{i,j,k,l=1}^{n} b_{mp,ij,kl} g_{mij} g_{pkl} + \sum_{i,j,k=1}^{n} c_{mp,ijk} g_{ijk} + \sum_{i,j=1}^{n} d_{mp,ij} g_{ij}, \tag{4.1.1}$$

with the fourth derivatives

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij} g_{mpij} = &\frac{1}{I^{3/2}} \sum_{i,j=1}^{n} (g_j^2 + Igg_{jj} - g^3 g_j \sum_{k=1}^{n} g_k g_{jk}) g_{mpii} \\ &- \frac{1}{I^{3/2}} \sum_{i,j=1}^{n} (g_i g_j + Igg_{ij} + g^3 g_i g_j \sum_{k=1}^{n} g_{kk} - 2g^3 g_i \sum_{k=1}^{n} g_k g_{jk}) g_{mpij},\end{aligned} \tag{4.1.2}$$

the terms which are quadratic in third derivatives

$$\sum_{i,j,k,l=1}^{n} b_{mp,ij,kl} g_{mij} g_{pkl} = \frac{1}{\sqrt{I}} g \sum_{i=1}^{n} g_{mii} \sum_{j=1}^{n} g_{pjj} - \frac{1}{\sqrt{I}} g \sum_{i,j=1}^{n} g_{mij} g_{pij} + \frac{2}{I^{3/2}} g^3 \sum_{i,j,k=1}^{n} g_j g_k g_{mij} g_{pik}$$
$$- \frac{1}{I^{3/2}} g^3 \sum_{i=1}^{n} g_{mii} \sum_{k,l=1}^{n} g_k g_l g_{pkl} - \frac{1}{I^{3/2}} g^3 \sum_{i,j=1}^{n} g_i g_j g_{mij} \sum_{k=1}^{n} g_{pkk}, \tag{4.1.3}$$

the terms which are linear in third derivatives $\sum_{i,j,k=1}^{n} c_{mp,ijk} g_{ijk}$ *for* $c_{mp,ijk} = c_{mp,ijk}(g, Dg, D^2 g)$

$$c_{mp,ijk} g_{ijk} = \frac{1}{2I^{5/2}}\Big(4g_k(g_{ik}g_{jj} - g_{jk}g_{ij}) - Ig^3 g_i(g_{kk}g_{jj} - g_{kj}^2) + 6g^5(g_i g_l g_k g_{lk} g_{jj} - g_i g_l g_k g_{jl} g_{jk})\Big) g_{mpi}$$
$$+ \frac{1}{I^{5/2}} g_{pi}\Big(- Ig^3 g_i(g_{kk}g_{mjj} - g_{kj}g_{mkj}) - 2Ig^3 g_k(g_{ik}g_{mjj} + g_{jj}g_{mik} - g_{jk}g_{mij} - g_{ij}g_{mjk})$$
$$+ 3g^5 g_i g_l g_k(g_{lk}g_{mjj} + g_{jj}g_{mlk} - 2g_{jl}g_{mjk})\Big)$$
$$+ \frac{1}{I^{5/2}} g_{mi}\Big(- Ig^3 g_i(g_{kk}g_{jjp} - g_{kj}g_{kjp}) - 2Ig^3 g_k(g_{ik}g_{jjp} + g_{jj}g_{ikp} - g_{jk}g_{ijp} - g_{ij}g_{jkp})$$
$$+ 3g^5 g_i g_l g_k(g_{lk}g_{jjp} + g_{jj}g_{lkp} - 2g_{jl}g_{jkp})\Big) + \frac{1}{I^{5/2}}\Big(2I(g_i g_{jj} - g_j g_{ij}) - 3g^2(g_i g_k^2 g_{jj} - g_i g_j g_k g_{jk})\Big) g_{imp}$$
$$+ \frac{1}{I^{5/2}} g_{pi}\Big(2I(g_i g_{jjm} - g_j g_{ijm}) - 3g^2 g_i(g_k^2 g_{jjm} - g_j g_k g_{jkm})\Big)$$
$$+ \frac{1}{I^{5/2}} g_{mi}\Big(2I(g_i g_{jjp} - g_j g_{ijp}) - 3g^2 g_i(g_k^2 g_{jjp} - g_j g_k g_{jkp})\Big)$$
$$+ \frac{1}{I^{5/2}} g_p\Big(I(g_{jj}g_{iim} - g_{ij}g_{ijm}) - 3g^2 g_i g_k(g_{ik}g_{jjm} + g_{jj}g_{ikm} - 2g_{jk}g_{ijm})\Big)$$
$$+ \frac{1}{I^{5/2}} g_m\Big(I(g_{jj}g_{iip} - g_{ij}g_{ijp}) - 3g^2 g_i g_k(g_{ik}g_{jjp} + g_{jj}g_{ikp} - 2g_{jk}g_{ijp})\Big)$$
$$- \frac{3}{I^{5/2}} g g_p g_k^2(g_j^2 g_{iim} - g_i g_j g_{ijm}) - \frac{3}{I^{5/2}} g g_m g_k^2(g_j^2 g_{iip} - g_i g_j g_{ijp})$$

and the terms involving only first-order and second-order derivatives $\sum_{i,j=1}^{n} d_{mp,ij} g_{ij}$ *for* $d_{mp,ij} = d_{mp,ij}(g, Dg, D^2 g)$

$$d_{mp,ij} g_{ij} = \frac{1}{2I^{5/2}}\Big(I(g_{ii}g_{jj} - g_{ij}^2) - 6g^2 g_i g_k(g_{ik}g_{jj} - g_{jk}g_{ij}) - 6g g_k^2(g_i^2 g_{jj} - g_j g_i g_{ij})\Big) g_{mp}$$
$$+ \frac{1}{2I^{5/2}} g_{mi}\Big(- Ig^3(g_{kk}g_{jj} - g_{kj}^2) g_{ip} - 4Ig^3(g_{ik}g_{jj} - g_{jk}g_{ij}) g_{kp}$$
$$+ 6g^5 g_j g_k(g_{jk}g_{ll} - g_{jl}g_{kl}) g_{ip} - 2g^5 g_i g_j(g_{kk}g_{ll} - g_{kl}^2) g_{jp}$$
$$+ 4I(g_{jj}g_{ip} - g_{ij}g_{jp}) - 6g^2(g_k^2 g_{jj} - g_j g_k g_{jk}) g_{ip} - 24g^2 g_i g_j g_{kk} g_{pj}\Big)$$
$$+ \frac{5}{2I^{7/2}}\Big(Ig^3 g_i(g_{kk}g_{jj} - g_{kj}^2) - 6g^5 g_i(g_l g_k g_{lk} g_{jj} - g_l g_k g_{jl} g_{jk}) + 6g^2 g_i(g_k^2 g_{jj} - g_j g_k g_{jk})\Big) g^2 g_q g_{mi} g_{pq}$$
$$+ \frac{6}{I^{5/2}} g^2 g_k g_{jk} g_i g_{pi} g_{mj} + \frac{6}{I^{5/2}} g^2 g_k g_{jk} g_i g_{mi} g_{pj}$$
$$+ \frac{6}{I^{5/2}} g^5 g_k g_l(g_{ki}g_{jj} - g_{ji}g_{jk}) g_{pi} g_{ml} + \frac{6}{I^{5/2}} g^5 g_k g_l(g_{ki}g_{jj} - g_{ji}g_{jk}) g_{mi} g_{pl}$$

$$
\begin{aligned}
&-\frac{3}{2I^{5/2}}\Big(g^2g_i(g_{kk}g_{jj}-g_{kj}^2)+4g^2g_k(g_{ik}g_{jj}-g_{jk}g_{ij})\Big)g_pg_{mi}+\frac{15}{I^{7/2}}g^4g_i(g_lg_kg_{lk}g_{jj}-g_lg_kg_{jl}g_{jk})g_pg_{mi}\\
&-\frac{3}{2I^{5/2}}\Big(g^2g_i(g_{kk}g_{jj}-g_{kj}^2)+4g^2g_k(g_{ik}g_{jj}-g_{jk}g_{ij})\Big)g_mg_{pi}+\frac{15}{I^{7/2}}g^4g_i(g_lg_kg_{lk}g_{jj}-g_lg_kg_{jl}g_{jk})g_mg_{pi}\\
&-\frac{6}{I^{5/2}}\Big(gg_k^2(g_ig_{jj}-g_jg_{ij})+gg_i(g_k^2g_{jj}-g_jg_kg_{jk})\Big)g_pg_{mi}+\frac{15}{I^{7/2}}g^3g_l^2g_i(g_k^2g_{jj}-g_jg_kg_{jk})g_pg_{mi}\\
&-\frac{6}{I^{5/2}}\Big(gg_k^2(g_ig_{jj}-g_jg_{ij})+gg_i(g_k^2g_{jj}-g_jg_kg_{jk})\Big)g_mg_{pi}+\frac{15}{I^{7/2}}g^3g_l^2g_i(g_k^2g_{jj}-g_jg_kg_{jk})g_mg_{pi}\\
&+\frac{1}{I^{5/2}}\Big(-6g(g_{ik}g_{jj}-g_{jk}g_{ij})g_ig_k+g(g_{ii}g_{jj}-g_{ij}^2)g_k^2-3g_k^2(g_i^2g_{jj}-g_jg_ig_{ij})\Big)g_mg_p\\
&-\frac{5}{2I^{7/2}}\Big(I(g_{ii}g_{jj}-g_{ij}^2)-6g^2g_ig_k(g_{ik}g_{jj}-g_{jk}g_{ij})-6gg_k^2(g_i^2g_{jj}-g_jg_ig_{ij})\Big)gg_l^2g_mg_p.
\end{aligned}
$$

Proof. Using Lemma 3.1.1 about the evolution of the first-order derivatives of g, we have

$$
(g_{mp})_t=L[g_{mp}]+cg_{mp}+\sum_{i,j=1}^{n}\partial_p(a_{ij})g_{mij}+\sum_{i=1}^{n}\partial_p(b_i)g_{mi}+\partial_p(c)g_m. \tag{4.1.4}
$$

Then the equation (4.1.1) is obtained by direct calculations of differentiating the coefficients a_{ij} (3.1.2) and b_i (3.1.3) of the operator L (3.1.1) and c (3.1.4) with respect to x_p. Note that, a_{ij} of 4.1.2 is equal to a_{ij} (3.1.2) in L (3.1.1).

Indeed, first

$$
\begin{aligned}
\partial_p(a_{ij})g_{mij}&=-\frac{3}{2I^{5/2}}I_p(Igg_{jj}-g^3g_jg_kg_{jk}+g_j^2)g_{iim}+\frac{1}{I^{3/2}}(Igg_{jj}-g^3g_jg_kg_{jk}+g_j^2)_pg_{iim}\\
&+\frac{3}{2I^{5/2}}I_p(Igg_{ij}+g^3g_ig_jg_{kk}-2g^3g_ig_kg_{jk}+g_ig_j)g_{ijm}\\
&-\frac{1}{I^{3/2}}(Igg_{ij}+g^3g_ig_jg_{kk}-2g^3g_ig_kg_{jk}+g_ig_j)_pg_{ijm}\\
&=\frac{1}{I^{3/2}}(Igg_{jjp}-g^3g_jg_kg_{jkp})g_{iim}-\frac{1}{I^{3/2}}Igg_{ijp}g_{ijm}+\frac{1}{I^{3/2}}g^3g_k(g_jg_{ikp}+g_ig_{jkp})g_{ijm}-\frac{1}{I^{3/2}}g^3g_ig_jg_{kkp}g_{ijm}\\
&-\frac{3}{I^{5/2}}(Igg_{jj}-g^3g_jg_kg_{jk}+g_j^2)g^2g_lg_{lp}g_{iim}-\frac{2}{I^{3/2}}g^2g_lgg_{ij}g_{lp}g_{ijm}\\
&+\frac{1}{I^{3/2}}(2g^2g_lgg_{jj}g_{lp}-g^3g_jg_{jk}g_{kp}-g^3g_kg_{jk}g_{jp}+2g_jg_{jp})g_{iim}\\
&+\frac{3}{I^{5/2}}(Igg_{ij}+g^3g_ig_jg_{kk}-g^3g_ig_kg_{jk}-g^3g_jg_kg_{ik}+g_ig_j)g^2g_lg_{lp}g_{ijm}-\frac{1}{I^{3/2}}(g_jg_{ip}+g_ig_{jp})g_{ijm}\\
&-\frac{1}{I^{3/2}}g^3g_{kk}(g_jg_{ip}+g_ig_{jp})g_{ijm}+\frac{1}{I^{3/2}}g^3\Big((g_jg_{ik}+g_ig_{jk})g_{kp}+g_k(g_{jk}g_{ip}+g_{ik}g_{jp})\Big)g_{ijm}\\
&-\frac{3}{I^{5/2}}(Igg_{jj}-g^3g_jg_kg_{jk}+g_j^2)gg_l^2g_pg_{iim}+\frac{1}{I^{3/2}}(2gg_l^2gg_{jj}+Ig_{jj}-3g^2g_jg_kg_{jk})g_pg_{iim}\\
&+\frac{3}{I^{5/2}}\Big(Igg_{ij}+g^3g_ig_jg_{kk}-g^3g_k(g_jg_{ik}+g_ig_{jk})+g_ig_j\Big)gg_l^2g_pg_{ijm}\\
&-\frac{1}{I^{3/2}}(2g^2g_l^2+I)g_{ij}g_pg_{ijm}-\frac{3}{I^{3/2}}g^2g_ig_jg_{kk}g_pg_{ijm}+\frac{3}{I^{3/2}}g^2g_k(g_jg_{ik}+g_ig_{jk})g_pg_{ijm}.
\end{aligned}
$$

Second,

$$
\begin{aligned}
\partial_p(b_i)g_{mi} = & -\frac{5}{2I^{7/2}}(g^2g_qg_{qp}+gg_q^2g_p)\Big(4I(g_ig_{jj}-g_jg_{ij})\\
& -Ig^2\Big(gg_i(g_{kk}g_{jj}-g_{kj}^2)+4gg_k(g_{ik}g_{jj}-g_{jk}g_{ij})\Big)-g^2(6g_ig_k^2g_{jj}-6g_ig_jg_kg_{jk})\\
& +6g^5(g_ig_lg_kg_{lk}g_{jj}-g_ig_lg_kg_{jl}g_{jk})\Big)g_{mi}\\
& +\frac{1}{2I^{5/2}}\Big(8(g^2g_kg_{kp}+gg_k^2g_p)(g_ig_{jj}-g_jg_{ij})+4I(g_ig_{jjp}-g_jg_{ijp}+g_{jj}g_{ip}-g_{ij}g_{jp})\\
& -2\Big((g^2g_lg_{lp}+gg_l^2g_p)g^2+Igg_p\Big)\Big(gg_i(g_{kk}g_{jj}-g_{kj}^2)+4gg_k(g_{ik}g_{jj}-g_{jk}g_{ij})\Big)\\
& -Ig^2\Big((gg_{ip}+g_ig_p)(g_{kk}g_{jj}-g_{kj}^2)+gg_i(g_{kk}g_{jjp}+g_{jj}g_{kkp}-2g_{kj}g_{kjp})\Big)\\
& -Ig^2\Big(4(gg_{kp}+g_kg_p)(g_{ik}g_{jj}-g_{jk}g_{ij})+4gg_k(g_{ik}g_{jjp}-g_{jk}g_{ijp}+g_{jj}g_{ikp}-g_{ij}g_{jkp})\Big)\\
& -12gg_p(g_ig_k^2g_{jj}-g_ig_jg_kg_{jk})-6g^2g_{ip}(g_k^2g_{jj}-g_jg_kg_{jk})\\
& -6g^2g_i(g_k^2g_{jjp}-g_jg_kg_{jkp}+2g_kg_{jj}g_{kp}-g_jg_{kp}g_{jk}-g_kg_{jp}g_{jk})\\
& +30g^4g_p(g_ig_lg_kg_{lk}g_{jj}-g_ig_lg_kg_{jl}g_{jk})+6g^5(g_ig_lg_{kp}+g_ig_kg_{lp}+g_lg_kg_{ip})(g_{lk}g_{jj}-g_{jl}g_{jk})\\
& +6g^5g_ig_lg_k(g_{lk}g_{jjp}-g_{jl}g_{jkp}+g_{jj}g_{lkp}-g_{jk}g_{jlp})\Big)g_{mi}\\
= & \frac{1}{2I^{5/2}}g_{mi}\Big(4I(g_ig_{jjp}-g_jg_{ijp})-2Ig^3g_i(g_{kk}g_{jjp}-g_{kj}g_{kjp})\\
& -4Ig^2gg_k(g_{ik}g_{jjp}+g_{jj}g_{ikp}-g_{jk}g_{ijp}-g_{ij}g_{jkp})-6g^2g_i(g_k^2g_{jjp}-g_jg_kg_{jkp})\\
& +6g^5g_ig_lg_k(g_{lk}g_{jjp}+g_{jj}g_{lkp}-g_{jl}g_{jkp}-g_{jk}g_{jlp})\Big)\\
& +\frac{1}{2I^{5/2}}g_{mi}\Big(4I(g_{jj}g_{ip}-g_{ij}g_{jp})-Ig^2(g_{kk}g_{jj}-g_{kj}^2)gg_{ip}-4Ig^2(g_{ik}g_{jj}-g_{jk}g_{ij})gg_{kp}\\
& -2\Big(gg_i(g_{kk}g_{jj}-g_{kj}^2)+4gg_k(g_{ik}g_{jj}-g_{jk}g_{ij})\Big)g^4g_lg_{lp}\\
& +6g^5(g_{lk}g_{jj}-g_{jl}g_{jk})(g_ig_lg_{kp}+g_ig_kg_{lp}+g_lg_kg_{ip})\\
& -6g^2(g_k^2g_{jj}-g_jg_kg_{jk})g_{ip}+8(g_ig_{jj}-g_jg_{ij})g^2g_kg_{kp}-12g^2g_i(g_jg_{kk}g_{jp}-g_kg_{jk}g_{jp})\Big)\\
& -\frac{5}{2I^{7/2}}g_{mi}\Big(4I(g_ig_{jj}-g_jg_{ij})\\
& -Ig^2\Big(gg_i(g_{kk}g_{jj}-g_{kj}^2)+4gg_k(g_{ik}g_{jj}-g_{jk}g_{ij})\Big)-6g^2g_i(g_k^2g_{jj}-g_jg_kg_{jk})\\
& +6g^5g_i(g_lg_kg_{lk}g_{jj}-g_lg_kg_{jl}g_{jk})\Big)g^2g_qg_{qp}\\
& +\frac{1}{2I^{5/2}}g_{mi}\Big(8(g_ig_{jj}-g_jg_{ij})gg_k^2-2\Big(gg_i(g_{kk}g_{jj}-g_{kj}^2)+4gg_k(g_{ik}g_{jj}-g_{jk}g_{ij})\Big)(g^3g_l^2+Ig)\\
& -Ig^2(g_{kk}g_{jj}-g_{kj}^2)g_i-4Ig^2(g_{ik}g_{jj}-g_{jk}g_{ij})g_k-12gg_i(g_k^2g_{jj}-g_jg_kg_{jk})\\
& +30g^4g_i(g_lg_kg_{lk}g_{jj}-g_lg_kg_{jl}g_{jk})\Big)g_p\\
& -\frac{5}{2I^{7/2}}g_{mi}\Big(4I(g_ig_{jj}-g_jg_{ij})\\
& -Ig^2\Big(gg_i(g_{kk}g_{jj}-g_{kj}^2)+4gg_k(g_{ik}g_{jj}-g_{jk}g_{ij})\Big)-6g^2g_i(g_k^2g_{jj}-g_jg_kg_{jk})\\
& +6g^5g_i(g_lg_kg_{lk}g_{jj}-g_lg_kg_{jl}g_{jk})\Big)gg_q^2g_p.
\end{aligned}
$$

And third,

$$\partial_p(c)g_m = -\frac{5}{4I^{7/2}}I_p\Big(I(g_{ii}g_{jj}-g_{ij}^2)-6g^2g_ig_k(g_{ik}g_{jj}-g_{jk}g_{ij})-6gg_k^2(g_i^2g_{jj}-g_jg_ig_{ij})\Big)g_m$$
$$+\frac{1}{2I^{5/2}}\Big(I(g_{ii}g_{jj}-g_{ij}^2)-6g^2g_ig_k(g_{ik}g_{jj}-g_{jk}g_{ij})-6gg_k^2(g_i^2g_{jj}-g_jg_ig_{ij})\Big)_p g_m$$
$$=-\frac{5}{2I^{7/2}}g_m(g^2g_lg_{lp}+gg_pg_l^2)\Big(I(g_{ii}g_{jj}-g_{ij}^2)-6g^2g_ig_k(g_{ik}g_{jj}-g_{jk}g_{ij})-6gg_k^2(g_i^2g_{jj}-g_jg_ig_{ij})\Big)$$
$$+\frac{1}{2I^{5/2}}g_m\Big(2(g_{ii}g_{jj}-g_{ij}^2)(g^2g_kg_{kp}+gg_pg_k^2)+I(g_{ii}g_{jjp}+g_{jj}g_{iip}-2g_{ij}g_{ijp})$$
$$-6(g^2g_ig_{kp}+g^2g_kg_{ip}+2gg_ig_kg_p)(g_{ik}g_{jj}-g_{jk}g_{ij})$$
$$-6g^2g_ig_k(g_{ik}g_{jjp}-g_{jk}g_{ijp}+g_{jj}g_{ikp}-g_{ij}g_{jkp})$$
$$-6(2gg_kg_{kp}+g_k^2g_p)(g_i^2g_{jj}-g_jg_ig_{ij})-6gg_k^2(g_i^2g_{jjp}+2g_ig_{jj}g_{ip}-g_jg_ig_{ijp}-g_jg_{ij}g_{ip}-g_ig_{ij}g_{jp})\Big)$$
$$=\frac{1}{I^{5/2}}g_m\Big(I(g_{jj}g_{iip}-g_{ij}g_{ijp})-3g^2g_ig_k(g_{ik}g_{jjp}-g_{jk}g_{ijp}+g_{jj}g_{ikp}-g_{ij}g_{jkp})-3gg_k^2(g_i^2g_{jjp}-g_jg_ig_{ijp})\Big)$$
$$+\frac{1}{I^{5/2}}g_m\Big(-3(g_{ik}g_{jj}-g_{jk}g_{ij})(g^2g_ig_{kp}+g^2g_kg_{ip})+(g_{ii}g_{jj}-g_{ij}^2)g^2g_kg_{kp}$$
$$-3(g_i^2g_{jj}-g_jg_ig_{ij})2gg_kg_{kp}-3gg_k^2(2g_ig_{jj}g_{ip}-g_jg_{ij}g_{ip}-g_ig_{ij}g_{jp})\Big)$$
$$-\frac{5}{2I^{7/2}}g_m\Big(I(g_{ii}g_{jj}-g_{ij}^2)-6g^2g_ig_k(g_{ik}g_{jj}-g_{jk}g_{ij})-6gg_k^2(g_i^2g_{jj}-g_jg_ig_{ij})\Big)g^2g_lg_{lp}$$
$$+\frac{1}{I^{5/2}}g_m\Big(-6(g_{ik}g_{jj}-g_{jk}g_{ij})gg_ig_k+(g_{ii}g_{jj}-g_{ij}^2)gg_k^2-3(g_i^2g_{jj}-g_jg_ig_{ij})g_k^2\Big)g_p$$
$$-\frac{5}{2I^{7/2}}g_m\Big(I(g_{ii}g_{jj}-g_{ij}^2)-6g^2g_ig_k(g_{ik}g_{jj}-g_{jk}g_{ij})-6gg_k^2(g_i^2g_{jj}-g_jg_ig_{ij})\Big)gg_l^2g_p.$$

Now, they sum up to

$$\partial_p(a_{ij})g_{mij}+\partial_p(b_i)g_{mi}+\partial_p(c)g_m$$
$$=\frac{1}{I^{3/2}}(Igg_{jjp}-g^3g_jg_kg_{jkp})g_{iim}-\frac{1}{I^{3/2}}Igg_{ijp}g_{ijm}+\frac{1}{I^{3/2}}g^3g_k(g_jg_{ikp}+g_ig_{jkp})g_{ijm}-\frac{1}{I^{3/2}}g^3g_ig_jg_{kkp}g_{ijm}$$
$$-\frac{3}{I^{5/2}}(Igg_{jj}-g^3g_jg_kg_{jk}+g_j^2)g^2g_lg_{lp}g_{iim}-\frac{2}{I^{3/2}}g^2g_lgg_{ij}g_{lp}g_{ijm}$$
$$+\frac{1}{I^{3/2}}(2g^2g_lgg_{jj}g_{lp}-g^3g_jg_{jk}g_{kp}-g^3g_kg_{jk}g_{jp}+2g_jg_{jp})g_{iim}$$
$$+\frac{3}{I^{5/2}}(Igg_{ij}+g^3g_ig_jg_{kk}-g^3g_ig_kg_{jk}-g^3g_jg_kg_{ik}+g_ig_j)g^2g_lg_{lp}g_{ijm}-\frac{1}{I^{3/2}}(g_jg_{ip}+g_ig_{jp})g_{ijm}$$
$$-\frac{1}{I^{3/2}}g^3g_{kk}(g_jg_{ip}+g_ig_{jp})g_{ijm}+\frac{1}{I^{3/2}}g^3\Big((g_jg_{ik}+g_ig_{jk})g_{kp}+g_k(g_{jk}g_{ip}+g_{ik}g_{jp})\Big)g_{ijm}$$
$$-\frac{3}{I^{5/2}}(Igg_{jj}-g^3g_jg_kg_{jk}+g_j^2)gg_l^2g_pg_{iim}+\frac{1}{I^{3/2}}(2gg_l^2gg_{jj}+Ig_{jj}-3g^2g_jg_kg_{jk})g_pg_{iim}$$
$$+\frac{3}{I^{5/2}}\Big(Igg_{ij}+g^3g_ig_jg_{kk}-g^3g_k(g_jg_{ik}+g_ig_{jk})+g_ig_j\Big)gg_l^2g_pg_{ijm}$$
$$-\frac{1}{I^{3/2}}(2g^2g_l^2+I)g_{ij}g_pg_{ijm}-\frac{3}{I^{3/2}}g^2g_ig_jg_{kk}g_pg_{ijm}+\frac{3}{I^{3/2}}g^2g_k(g_jg_{ik}+g_ig_{jk})g_pg_{ijm}$$
$$+\frac{1}{2I^{5/2}}g_{mi}\Big(4I(g_ig_{jjp}-g_jg_{ijp})-2Ig^3g_i(g_{kk}g_{jjp}-g_{kj}g_{kjp})$$

$$
\begin{aligned}
&- 4Ig^2 gg_k(g_{ik}g_{jjp} + g_{jj}g_{ikp} - g_{jk}g_{ijp} - g_{ij}g_{jkp}) - 6g^2 g_i(g_k^2 g_{jjp} - g_j g_k g_{jkp})\\
&+ 6g^5 g_i g_l g_k(g_{lk}g_{jjp} + g_{jj}g_{lkp} - g_{jl}g_{jkp} - g_{jk}g_{jlp})\Big)\\
&+ \frac{1}{I^{5/2}} g_m\Big(I(g_{jj}g_{iip} - g_{ij}g_{ijp}) - 3g^2 g_i g_k(g_{ik}g_{jjp} - g_{jk}g_{ijp} + g_{jj}g_{ikp} - g_{ij}g_{jkp}) - 3gg_k^2(g_i^2 g_{jjp} - g_j g_i g_{ijp})\Big)\\
&+ \frac{1}{2I^{5/2}} g_{mi}\Big(4I(g_{jj}g_{ip} - g_{ij}g_{jp}) - Ig^2(g_{kk}g_{jj} - g_{kj}^2)gg_{ip} - 4Ig^2(g_{ik}g_{jj} - g_{jk}g_{ij})gg_{kp}\\
&- 2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big)g^4 g_l g_{lp}\\
&+ 6g^5(g_{lk}g_{jj} - g_{jl}g_{jk})(g_i g_l g_{kp} + g_i g_k g_{lp} + g_l g_k g_{ip})\\
&- 6g^2(g_k^2 g_{jj} - g_j g_k g_{jk})g_{ip} + 8(g_i g_{jj} - g_j g_{ij})g^2 g_k g_{kp} - 12g^2 g_i(g_j g_{kk} g_{jp} - g_k g_{jk} g_{jp})\Big)\\
&- \frac{5}{2I^{7/2}} g_{mi}\Big(4I(g_i g_{jj} - g_j g_{ij}) - Ig^2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big) - 6g^2 g_i(g_k^2 g_{jj} - g_j g_k g_{jk})\\
&+ 6g^5 g_i(g_l g_k g_{lk} g_{jj} - g_l g_k g_{jl} g_{jk})\Big)g^2 g_q g_{qp}\\
&+ \frac{1}{2I^{5/2}} g_{mi}\Big(8(g_i g_{jj} - g_j g_{ij})gg_k^2 - 2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big)(g^3 g_l^2 + Ig)\\
&- Ig^2(g_{kk}g_{jj} - g_{kj}^2)g_i - 4Ig^2(g_{ik}g_{jj} - g_{jk}g_{ij})g_k - 12gg_i(g_k^2 g_{jj} - g_j g_k g_{jk})\\
&+ 30g^4 g_i(g_l g_k g_{lk} g_{jj} - g_l g_k g_{jl} g_{jk})\Big)g_p\\
&- \frac{5}{2I^{7/2}} g_{mi}\Big(4I(g_i g_{jj} - g_j g_{ij}) - Ig^2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big) - 6g^2 g_i(g_k^2 g_{jj} - g_j g_k g_{jk})\\
&+ 6g^5 g_i(g_l g_k g_{lk} g_{jj} - g_l g_k g_{jl} g_{jk})\Big)gg_q^2 g_p\\
&+ \frac{1}{I^{5/2}} g_m\Big(- 3(g_{ik}g_{jj} - g_{jk}g_{ij})(g^2 g_i g_{kp} + g^2 g_k g_{ip}) + (g_{ii}g_{jj} - g_{ij}^2)g^2 g_k g_{kp}\\
&- 3(g_i^2 g_{jj} - g_j g_i g_{ij})2gg_k g_{kp} - 3gg_k^2(2g_i g_{jj} g_{ip} - g_j g_{ij} g_{ip} - g_i g_{ij} g_{jp})\Big)\\
&- \frac{5}{2I^{7/2}} g_m\Big(I(g_{ii}g_{jj} - g_{ij}^2) - 6g^2 g_i g_k(g_{ik}g_{jj} - g_{jk}g_{ij}) - 6gg_k^2(g_i^2 g_{jj} - g_j g_i g_{ij})\Big)g^2 g_l g_{lp}\\
&+ \frac{1}{I^{5/2}} g_m\Big(- 6(g_{ik}g_{jj} - g_{jk}g_{ij})gg_i g_k + (g_{ii}g_{jj} - g_{ij}^2)gg_k^2 - 3(g_i^2 g_{jj} - g_j g_i g_{ij})g_k^2\Big)g_p\\
&- \frac{5}{2I^{7/2}} g_m\Big(I(g_{ii}g_{jj} - g_{ij}^2) - 6g^2 g_i g_k(g_{ik}g_{jj} - g_{jk}g_{ij}) - 6gg_k^2(g_i^2 g_{jj} - g_j g_i g_{ij})\Big)gg_l^2 g_p.
\end{aligned}
$$

Adding the terms above to the following ones

$$
\begin{aligned}
L[g_{mp}] + cg_{mp} &= \frac{1}{I^{3/2}}(Igg_{jj} - g^3 g_j g_k g_{jk} + g_j^2)g_{iimp} - \frac{1}{I^{3/2}}(Igg_{ij} + g^3 g_i g_j g_{kk} - 2g^3 g_i g_k g_{jk} + g_i g_j)g_{ijmp}\\
&+ \frac{1}{2I^{5/2}}\Big(4I(g_i g_{jj} - g_j g_{ij}) - Ig^2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big) - g^2(6g_i g_k^2 g_{jj} - 6g_i g_j g_k g_{jk})\\
&+ 6g^5(g_i g_l g_k g_{lk} g_{jj} - g_i g_l g_k g_{jl} g_{jk})\Big)g_{imp}\\
&+ \frac{1}{2I^{5/2}}\Big(I(g_{ii}g_{jj} - g_{ij}^2) - 6g^2 g_i g_k(g_{ik}g_{jj} - g_{jk}g_{ij}) - 6gg_k^2(g_i^2 g_{jj} - g_j g_i g_{ij})\Big)g_{mp},
\end{aligned}
\tag{4.1.5}
$$

we have

$$
\begin{aligned}
(g_{mp})_t &= L[g_{mp}] + cg_{mp} + \partial_p(a_{ij})g_{mij} + \partial_p(b_i)g_{mi} + \partial_p(c)g_m \\
&= \frac{1}{I^{3/2}}(Igg_{jj} - g^3 g_j g_k g_{jk} + g_j^2)g_{iimp} - \frac{1}{I^{3/2}}(Igg_{ij} + g^3 g_i g_j g_{kk} - 2g^3 g_i g_k g_{jk} + g_i g_j)g_{ijmp} \\
&+ \frac{1}{I^{3/2}}(Igg_{jjp} - g^3 g_j g_k g_{jkp})g_{iim} - \frac{1}{I^{3/2}} Igg_{ijp}g_{ijm} + \frac{1}{I^{3/2}} g^3 g_k (g_j g_{ikp} + g_i g_{jkp})g_{ijm} - \frac{1}{I^{3/2}} g^3 g_i g_j g_{kkp} g_{ijm} \\
&+ \frac{1}{2I^{5/2}}\Big(4I(g_i g_{jj} - g_j g_{ij}) - Ig^2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big) - g^2(6g_i g_k^2 g_{jj} - 6g_i g_j g_k g_{jk}) \\
&\quad + 6g^5(g_i g_l g_k g_{lk} g_{jj} - g_i g_l g_k g_{jl} g_{jk})\Big)g_{imp} \\
&- \frac{3}{I^{5/2}}(Igg_{jj} - g^3 g_j g_k g_{jk} + g_j^2)g^2 g_l g_{lp} g_{iim} - \frac{2}{I^{3/2}} g^2 g_l g g_{ij} g_{lp} g_{ijm} \\
&+ \frac{1}{I^{3/2}}(2g^2 g_l g g_{jj} g_{lp} - g^3 g_j g_{jk} g_{kp} - g^3 g_k g_{jk} g_{jp} + 2g_j g_{jp})g_{iim} \\
&+ \frac{3}{I^{5/2}}(Igg_{ij} + g^3 g_i g_j g_{kk} - g^3 g_i g_k g_{jk} - g^3 g_j g_k g_{ik} + g_i g_j)g^2 g_l g_{lp} g_{ijm} - \frac{1}{I^{3/2}}(g_j g_{ip} + g_i g_{jp})g_{ijm} \\
&- \frac{1}{I^{3/2}} g^3 g_{kk}(g_j g_{ip} + g_i g_{jp})g_{ijm} + \frac{1}{I^{3/2}} g^3\Big((g_j g_{ik} + g_i g_{jk})g_{kp} + g_k(g_{jk}g_{ip} + g_{ik}g_{jp})\Big)g_{ijm} \\
&- \frac{3}{I^{5/2}}(Igg_{jj} - g^3 g_j g_k g_{jk} + g_j^2)gg_l^2 g_p g_{iim} + \frac{1}{I^{3/2}}(2gg_l^2 gg_{jj} + Ig_{jj} - 3g^2 g_j g_k g_{jk})g_p g_{iim} \\
&+ \frac{3}{I^{5/2}}\Big(Igg_{ij} + g^3 g_i g_j g_{kk} - g^3 g_k(g_j g_{ik} + g_i g_{jk}) + g_i g_j\Big)gg_l^2 g_p g_{ijm} \\
&- \frac{1}{I^{3/2}}(2g^2 g_l^2 + I)g_{ij} g_p g_{ijm} - \frac{3}{I^{3/2}} g^2 g_i g_j g_{kk} g_p g_{ijm} + \frac{3}{I^{3/2}} g^2 g_k(g_j g_{ik} + g_i g_{jk})g_p g_{ijm} \\
&+ \frac{1}{2I^{5/2}} g_{mi}\Big(4I(g_i g_{jjp} - g_j g_{ijp}) - 2Ig^3 g_i(g_{kk}g_{jjp} - g_{kj}g_{kjp}) \\
&\quad - 4Ig^2 gg_k(g_{ik}g_{jjp} + g_{jj}g_{ikp} - g_{jk}g_{ijp} - g_{ij}g_{jkp}) - 6g^2 g_i(g_k^2 g_{jjp} - g_j g_k g_{jkp}) \\
&\quad + 6g^5 g_i g_l g_k(g_{lk}g_{jjp} + g_{jj}g_{lkp} - g_{jl}g_{jkp} - g_{jk}g_{jlp})\Big) \\
&+ \frac{1}{I^{5/2}} g_m\Big(I(g_{jj}g_{iip} - g_{ij}g_{ijp}) - 3g^2 g_i g_k(g_{ik}g_{jjp} - g_{jk}g_{ijp} + g_{jj}g_{ikp} - g_{ij}g_{jkp}) - 3gg_k^2(g_i^2 g_{jjp} - g_j g_i g_{ijp})\Big) \\
&+ \frac{1}{2I^{5/2}}\Big(I(g_{ii}g_{jj} - g_{ij}^2) - 6g^2 g_i g_k(g_{ik}g_{jj} - g_{jk}g_{ij}) - 6gg_k^2(g_i^2 g_{jj} - g_j g_i g_{ij})\Big)g_{mp} \\
&+ \frac{1}{2I^{5/2}} g_{mi}\Big(4I(g_{jj}g_{ip} - g_{ij}g_{jp}) - Ig^2(g_{kk}g_{jj} - g_{kj}^2)gg_{ip} - 4Ig^2(g_{ik}g_{jj} - g_{jk}g_{ij})gg_{kp} \\
&\quad - 2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big)g^4 g_l g_{lp} \\
&\quad + 6g^5(g_{lk}g_{jj} - g_{jl}g_{jk})(g_i g_l g_{kp} + g_i g_k g_{lp} + g_l g_k g_{ip}) \\
&\quad - 6g^2(g_k^2 g_{jj} - g_j g_k g_{jk})g_{ip} + 8(g_i g_{jj} - g_j g_{ij})g^2 g_k g_{kp} - 12g^2 g_i(g_j g_{kk} g_{jp} - g_k g_{jk} g_{jp})\Big) \\
&- \frac{5}{2I^{7/2}} g_{mi}\Big(4I(g_i g_{jj} - g_j g_{ij}) - Ig^2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big) - 6g^2 g_i(g_k^2 g_{jj} - g_j g_k g_{jk}) \\
&\quad + 6g^5 g_i(g_l g_k g_{lk} g_{jj} - g_l g_k g_{jl} g_{jk})\Big)g^2 g_q g_{qp} \\
&+ \frac{1}{2I^{5/2}} g_{mi}\Big(8(g_i g_{jj} - g_j g_{ij})gg_k^2 - 2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big)(g^3 g_l^2 + Ig)
\end{aligned}
$$

$$
\begin{aligned}
&- Ig^2(g_{kk}g_{jj} - g_{kj}^2)g_i - 4Ig^2(g_{ik}g_{jj} - g_{jk}g_{ij})g_k - 12gg_i(g_k^2 g_{jj} - g_j g_k g_{jk}) \\
&+ 30g^4 g_i(g_l g_k g_{lk} g_{jj} - g_l g_k g_{jl} g_{jk})\Big)g_p \\
&- \frac{5}{2I^{7/2}} g_{mi}\Big(4I(g_i g_{jj} - g_j g_{ij}) - Ig^2\Big(gg_i(g_{kk}g_{jj} - g_{kj}^2) + 4gg_k(g_{ik}g_{jj} - g_{jk}g_{ij})\Big) - 6g^2 g_i(g_k^2 g_{jj} - g_j g_k g_{jk}) \\
&+ 6g^5 g_i(g_l g_k g_{lk} g_{jj} - g_l g_k g_{jl} g_{jk})\Big)gg_q^2 g_p \\
&+ \frac{1}{I^{5/2}} g_m\Big(- 3(g_{ik}g_{jj} - g_{jk}g_{ij})(g^2 g_i g_{kp} + g^2 g_k g_{ip}) + (g_{ii}g_{jj} - g_{ij}^2)g^2 g_k g_{kp} \\
&- 3(g_i^2 g_{jj} - g_j g_i g_{ij})2gg_k g_{kp} - 3gg_k^2(2g_i g_{jj} g_{ip} - g_j g_{ij} g_{ip} - g_i g_{ij} g_{jp})\Big) \\
&- \frac{5}{2I^{7/2}} g_m\Big(I(g_{ii}g_{jj} - g_{ij}^2) - 6g^2 g_i g_k(g_{ik}g_{jj} - g_{jk}g_{ij}) - 6gg_k^2(g_i^2 g_{jj} - g_j g_i g_{ij})\Big)g^2 g_l g_{lp} \\
&+ \frac{1}{I^{5/2}} g_m\Big(- 6(g_{ik}g_{jj} - g_{jk}g_{ij})gg_i g_k + (g_{ii}g_{jj} - g_{ij}^2)gg_k^2 - 3(g_i^2 g_{jj} - g_j g_i g_{ij})g_k^2\Big)g_p \\
&- \frac{5}{2I^{7/2}} g_m\Big(I(g_{ii}g_{jj} - g_{ij}^2) - 6g^2 g_i g_k(g_{ik}g_{jj} - g_{jk}g_{ij}) - 6gg_k^2(g_i^2 g_{jj} - g_j g_i g_{ij})\Big)gg_l^2 g_p
\end{aligned}
$$

and this completes the proof.

□

4.2 Second-order derivative estimates

Lemma 4.2.1. *Let us assume the conditions in the subsection 1.4. Then there exists a constant $C > 0$ satisfying*

$$
\begin{aligned}
&0 < \sum_{i,j=1}^{n}(g_i^2 g_{jj} - g_i g_j g_{ij}) = g_\nu^2 \sum_{\tau:\text{tangential to } \Gamma_\varepsilon} g_{\tau\tau} = g_\nu^3 H_{\Gamma_\varepsilon} \le C, \\
&0 < \sum_{\tau:\text{tangential to } \Gamma_\varepsilon} g_{\tau\tau} \le C
\end{aligned}
\tag{4.2.1}
$$

on $\Gamma_\varepsilon(t)$ for each $0 \le \varepsilon \le 1$ and $0 \le t \le T$, where ν is the outer unit normal vector to the level set $\Gamma_\varepsilon(t)$, and H_{Γ_ε} is the mean curvature of the level set $\Gamma_\varepsilon(t)$. Hence the mean curvature of level sets is uniformly bounded above.

Proof. We may assume $0 \le g < \varepsilon$, for sufficiently small $0 < \varepsilon \le \varepsilon_0$, since H_{Γ_ε} is positive and bounded above on $\{\frac{\varepsilon}{2} \le g \le 1\}$ by the existence of a strictly smooth solution from [11]. Let us define a quantity

$$
X := \sum_{i,j=1}^{n}(g_i^2 g_{jj} - g_i g_j g_{ij}) + \Delta f = \sum_{i,j=1}^{n}(g_i^2 g_{jj} - g_i g_j g_{ij}) + \sum_{i=1}^{n}(g_i^2 + gg_{ii}). \tag{4.2.2}
$$

In Lemma 3.2.3 and 3.2.4, we have shown $|\nabla g|$ is bounded from above and below; there exist a constant $c > 0$ such that $0 < c \le |\nabla g| \le c^{-1}$ on $g > 0$, $0 \le t \le T < T_c$. Also, $\sum_{i,j=1}^{n}(g_i^2 g_{jj} - g_i g_j g_{ij})$ and Δf are positive by strict convexity from [11], and Δf is bounded above by (1.5.1). Hence, an upper bound of X will give an upper bound of both of the quantity $\sum_{i,j=1}^n (g_i^2 g_{jj} - g_i g_j g_{ij})$ and the tangential Laplacian $\sum_\tau g_{\tau\tau}$. We use the maximum principle to show that X is bounded above.

On the interface $\Gamma(t)$ where $g = 0$, we have

$$c \le g_t = \frac{1}{2\sqrt{I}} \sum_{i,j=1}^n \left(g_j^2 g_{ii} + g_i^2 g_{jj} - 2 g_i g_j g_{ij}\right) = \frac{X - \Delta f}{\sqrt{1 + g^2 |\nabla g|^2}} \le c^{-1} \tag{4.2.3}$$

for some constant $c > 0$ by Corollary 3.2.7, implying that $X \le C$ on $\Gamma(t)$. On the level set $\Gamma_1(t)$ where $g = 1$, $D_{ij} f = g_i g_j + g g_{ij} = g_i g_j + g_{ij}$ are bounded for all i, j, so X is bounded above on $\Gamma_1(t)$.

Hence, we can assume that X attains its space-time maximum at an interior point $P_0(t)$ of the domain for $0 \le t \le T < T_c$. At $P_0 \in \Gamma_{P_0}(t)$, we take the coordinate system where $g_i = 0$ for $i \ne 1$ and $e_1 = \nu$, the normal direction to $\Gamma_{P_0}(t)$, and $g_{ij} = 0$ for $i \ne j$. Then we have

$$X = g_1^2 \sum_{i=2}^n g_{ii} + (g_1^2 + g g_{11} + g \sum_{i=2}^n g_{ii}). \tag{4.2.4}$$

Differentiating X with respect to space variables at P_0, we get for all $1 \le k \le n$

$$0 = X_k = \sum_{i,j=1}^n (g_j^2 g_{iik} - g_i g_j g_{jik} + 2 g_j g_{ii} g_{jk} - 2 g_j g_{ji} g_{ik}) + \sum_{i=1}^n (g g_{iik} + 2 g_i g_{ik} + g_k g_{ii}). \tag{4.2.5}$$

We have, at P_0,

$$\begin{aligned} 0 = X_1 &= g g_{111} + (g_1^2 + g) \sum_{i \ne 1} g_{ii1} + 2 g_1 g_{11} \sum_{i \ne 1} g_{ii} + 3 g_1 g_{11} + g_1 \sum_{i \ne 1} g_{ii}, \\ 0 = X_k &= g_1^2 \sum_{i=1}^n g_{iik} - g_1^2 g_{11k} + \sum_{i=1}^n (g g_{iik}) = g g_{11k} + (g_1^2 + g) \sum_{i \ne 1} g_{iik}, \end{aligned} \tag{4.2.6}$$

and then

$$\sum_{i \ne 1} g_{ii1} = -\frac{1}{g + g_1^2} (g g_{111} + 2 g_1 g_{11} \sum_{i \ne 1} g_{ii} + 3 g_1 g_{11} + \sum_{i \ne 1} g_1 g_{ii}), \; \sum_{i \ne 1} g_{iik} = -\frac{1}{g + g_1^2} g g_{11k}, \text{ for } k \ne 1. \tag{4.2.7}$$

Differentiating X with respect to time, we get X_t as follows. We use the indices m, p instead of i, j in order to avoid any possible confusions. The right-hand side of the evolution equation of X will contain spatial derivatives of g of order no greater than 4, because the equation (4.1.1) is a second-order equation. Using an index $2 \le m \le n$, we can simply write the evolution of X as the following:

$$X_t = g_1^2 \sum_{m \ne 1} (g_{mm})_t + 2 g_1 (g_1)_t \sum_{m \ne 1} g_{mm} + \left(g (g_{11})_t + g \sum_{m \ne 1} (g_{mm})_t + 2 g_1 (g_1)_t + g_t g_{11} + g_t \sum_{m \ne 1} g_{mm}\right). \tag{4.2.8}$$

The second derivatives $g_{mm}, 1 \le m \le n$ evolve at P_0 by the equation (4.1.1) of $(g_{mm})_t$ with

$$
\begin{aligned}
\sum_{i,j,k=1}^{n} c_{mm,ijk} g_{ijk} &= \frac{1}{2I^{5/2}}\Big(4g_1(g_{11}g_{jj} - g_{j1}^2) - Ig^3 g_1(g_{kk}g_{jj} - g_{kj}^2) + 6g^5 g_1^3(g_{11}g_{jj} - g_{j1}^2)\Big)g_{mm1} \\
&+ \frac{2}{I^{5/2}} g_{m1}\Big(- Ig^3 g_1(g_{kk}g_{mjj} - g_{kj}g_{mkj}) - 2Ig^3 g_1 \sum_{j\neq 1}(g_{11}g_{mjj} + g_{jj}g_{m11}) + 3g^5 g_1^2 \sum_{j\neq 1}(g_{11}g_{mjj} + g_{jj}g_{m11})\Big) \\
&+ \frac{1}{I^{5/2}}(2g_1 - g^2 g_1^3)\sum_{j\neq 1} g_{jj}g_{mm1} + \frac{2}{I^{5/2}} g_{mi}\Big(2I(g_i g_{mjj} - g_1 g_{mi1}) - 3g^2 g_i(g_1^2 g_{mjj} - g_1^2 g_{m11})\Big) \\
&+ \frac{2}{I^{5/2}} g_m\Big(I(g_{jj}g_{mii} - g_{ij}g_{mij}) - 3g^2 g_1^2 \sum_{j\neq 1}(g_{11}g_{mjj} + g_{jj}g_{m11})\Big) - \frac{6}{I^{5/2}} g g_m g_1^4 \sum_{j\neq 1} g_{mjj}, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i,j=1}^{n} d_{mm,ij} g_{ij} &= \frac{1}{2I^{5/2}}\Big(I(g_{ii}g_{jj} - g_{ij}^2) - 6g^2 g_1^2(g_{11}g_{jj} - g_{j1}^2) - 6gg_1^4(g_{jj} - g_{11})\Big)g_{mm} \\
&+ \frac{1}{2I^{5/2}} g_{mi}\Big(- Ig^3(g_{kk}g_{jj} - g_{kj}^2)g_{mi} - 4Ig^3(g_{ii}g_{jj} - g_{ij}^2)g_{mi} + 6g^5 g_1^2(g_{11}g_{jj} - g_{j1}^2)g_{mi} \\
&- 2g^5 g_i g_1(g_{jj}g_{kk} - g_{jk}^2)g_{m1} + 4I(g_{jj}g_{mi} - g_{ij}g_{mj}) - 6g^2 g_1^2(g_{jj} - g_{11})g_{mi} - 24g^2 g_i g_1 g_{kk} g_{m1}\Big) \\
&+ \frac{5}{2I^{7/2}}\Big(Ig^3(g_{kk}g_{jj} - g_{kj}^2) - 6g^5 g_1^2(g_{11}g_{jj} - g_{j1}^2) + 6g^2 g_1^2(g_{jj} - g_{11})\Big)g^2 g_1^2 g_{m1}^2 \\
&+ \frac{12}{I^{5/2}} g^2 g_1^2 g_{11} g_{m1}^2 + \frac{12}{I^{5/2}} g^5 g_1^2(g_{11}g_{jj} - g_{11}^2)g_{m1}^2 \\
&- \frac{3}{I^{5/2}}\Big(g^2 g_1(g_{kk}g_{jj} - g_{kj}^2) + 4g^2 g_1(g_{11}g_{jj} - g_{11}^2)\Big)g_m g_{m1} + \frac{30}{I^{7/2}} g^4 g_1^3(g_{11}g_{jj} - g_{j1}^2)g_m g_{m1} \\
&- \frac{24}{I^{5/2}} g g_1^3(g_{jj} - g_{11})g_m g_{m1} + \frac{30}{I^{7/2}} g^3 g_1^5(g_{jj} - g_{11})g_m g_{m1} \\
&+ \frac{1}{I^{5/2}}\Big(- 6gg_1^2(g_{11}g_{jj} - g_{j1}^2) + gg_1^2(g_{ii}g_{jj} - g_{ij}^2) - 3g_1^4(g_{jj} - g_{11})\Big)g_m^2 \\
&- \frac{5}{2I^{7/2}}\Big(I(g_{ii}g_{jj} - g_{ij}^2) - 6g^2 g_1^2(g_{11}g_{jj} - g_{j1}^2) - 6gg_1^4(g_{jj} - g_{11})\Big)g g_1^2 g_m^2.
\end{aligned}
\tag{4.2.9}
$$

Now, let us consider the second-order derivatives of X. They are

$$
\begin{aligned}
X_{kl} &= \sum_{i,j=1}^{n}(g_j^2 g_{iikl} - g_i g_j g_{ijkl}) + \sum_{i=1}^{n}(gg_{iikl} + g_k g_{iil} + g_l g_{iik} + 2g_i g_{ikl} + g_{kl}g_{ii} + 2g_{ik}g_{il}) \\
&\quad + 2\sum_{i,j=1}^{n}(g_j g_{jl} g_{iik} - g_j g_{il} g_{ijk} + g_j g_{jk} g_{iil} - g_j g_{ik} g_{ijl} + g_j g_{ii} g_{jkl} - g_j g_{ji} g_{ikl} + g_{jk} g_{jl} g_{ii} - g_{jk} g_{il} g_{ij}).
\end{aligned}
\tag{4.2.10}
$$

At P_0 under the coordinates $g_k = 0$ for $k \neq 1$ and $g_{ij} = 0$ for $i \neq j$,

$$\begin{aligned} X_{kl} = {} & g_1^2 \sum_{i=2}^n g_{iikl} + 2g_1 g_{1l} \sum_{i=1}^n g_{iik} + 2g_1 g_{1k} \sum_{i=1}^n g_{iil} + 2g_1 \sum_{i\neq 1,k,l} g_{ii} g_{1kl} + 2g_{kl} g_{kk} \sum_{i\neq k} g_{ii} \\ & + (g \sum_{i=1}^n g_{iikl} + 2g_1 g_{1kl} + g_k \sum_{i=1}^n g_{iil} + g_l \sum_{i=1}^n g_{iik} + g_{kl} \sum_{i=1}^n g_{ii} + 2g_{kl} g_{kk}). \end{aligned} \tag{4.2.11}$$

In our coordinate system at P_0, the quantity $J = |\nabla g|^2 + g$ becomes $J = g_1^2 + g$. Then the term $\frac{J}{g}$ goes to $+\infty$ as g tends to 0^+ and $\Delta f = g_1^2 + g g_{11} + g \sum_{i\neq 1} g_{ii}$ is bounded so that $0 \leq f_{11} = g g_{11} + g_1^2 \leq C$ and $0 \leq f_{ii} = g g_{ii} \leq C$ for $i \neq 1$. At P_0, $0 = X_m = \sum_{i\neq 1}(g_1^2 + g) g_{iim} + g g_{11m}$ for $m \neq 1$ and $0 = X_1 = (g_1^2 + g) \sum_{i\neq 1} g_{ii1} + g g_{111} + g_1(1 + 2g_{11}) \sum_{i\neq 1} g_{ii} + 3g_1 g_{11}$, so that

$$J \sum_{i\neq 1} g_{ii1} = -\Big(g g_{111} + (g_1 + 2g_1 g_{11}) \sum_{i\neq 1} g_{ii} + 3g_1 g_{11}\Big), \ J \sum_{i\neq 1} g_{iik} = -g g_{11k} \text{ for } k \neq 1. \tag{4.2.12}$$

We use the finite and non-degenerate speed of the level set of g to bound the term $\Big(\sum_{i\neq 1} \frac{g_1^2}{g} g_{ii}\Big)^2$ above. Previously, we obtained the result that for some sufficiently small constant $0 < C < 1$,

$$C \leq \frac{g_t}{g_1} = \frac{1}{2g_1\sqrt{J}} g \sum_{2\leq i,j} (g_{ii} g_{jj} - g_{ij}^2) + \frac{1}{g_1 J^{3/2}} (g g_{11} \sum_{i\neq 1} g_{ii} + g_1^2 \sum_{i\neq 1} g_{ii}) \leq C^{-1}. \tag{4.2.13}$$

and we also see that, by the boundedness of g_1,

$$\begin{gathered} C \leq \frac{1}{2\sqrt{J}} g \sum_{2\leq i,j} (g_{ii} g_{jj} - g_{ij}^2) + \frac{1}{J^{3/2}} (g g_{11} \sum_{i\neq 1} g_{ii} + g_1^2 \sum_{i\neq 1} g_{ii}) \leq C^{-1}, \\ \frac{1}{2\sqrt{J}} g R_{g,2} + \frac{1}{J^{3/2}} g_1^2 \sum_{i\neq 1} g_{ii} \leq C^{-1}, \ \frac{1}{2J^{3/2}} g R_{g,2} + \frac{1}{J^{3/2}} g_1^2 \sum_{i\neq 1} g_{ii} \leq C^{-1}, \ 0 \leq \frac{1}{J^{3/2}} (g g_{11} + g_1^2) \sum_{i\neq 1} g_{ii} \leq C^{-1}. \end{gathered} \tag{4.2.14}$$

Since Δf is bounded and $X = g_1^2 \sum_{i\neq 1} g_{ii} + \Delta f$ in our coordinates at P_0, without loss of generality we may assume that

$$C^{-1} \leq \frac{1}{2J^{3/2}} g_1^2 \sum_{i\neq 1} g_{ii} \leq X \leq 2g_1^2 \sum_{i\neq 1} g_{ii}, \tag{4.2.15}$$

because, if $\frac{1}{2J^{3/2}} g_1^2 \sum_{i\neq 1} g_{ii} \leq C^{-1}$, then X is bounded above and we get the upper bound of $\sum_{i\neq 1} g_{ii}$, which is our goal. So we have

$$\begin{aligned} & 0 \leq \frac{1}{J^{3/2}} g_1^2 \sum_{i\neq 1} g_{ii} \leq C^{-1} - \frac{1}{2\sqrt{J}} g \sum_{2\leq i,j} (g_{ii} g_{jj} - g_{ij}^2) - \frac{1}{J^{3/2}} g g_{11} \sum_{i\neq 1} g_{ii} \text{ and} \\ & \frac{1}{2\sqrt{J}} g \sum_{2\leq i,j} (g_{ii} g_{jj} - g_{ij}^2) + \frac{1}{J^{3/2}} g g_{11} \sum_{i\neq 1} g_{ii} \leq C^{-1} - \frac{1}{J^{3/2}} g_1^2 \sum_{i\neq 1} g_{ii} \leq -\frac{1}{2J^{3/2}} g_1^2 \sum_{i\neq 1} g_{ii}, \end{aligned} \tag{4.2.16}$$

$$
\begin{aligned}
&0 \le \frac{1}{2I^{3/2}} g_1^2 \sum_{i\neq 1} g_{ii} \le -\frac{1}{2\sqrt{I}} g \sum_{2\le i,j} (g_{ii}g_{jj} - g_{ij}^2) - \frac{1}{I^{3/2}} g g_{11} \sum_{i\neq 1} g_{ii},\\
&0 \le \frac{g_1^2}{g} \sum_{i\neq 1} g_{ii} \le -I \sum_{2\le i,j} (g_{ii}g_{jj} - g_{ij}^2) - 2g_{11} \sum_{i\neq 1} g_{ii} = -IR_{g,2} + 2g^2 g_1^2 g_{11} \sum_{i\neq 1} g_{ii} \text{ and} \qquad (4.2.17)\\
&2g_{11} \sum_{i\neq 1} g_{ii} \le -I \sum_{2\le i,j} (g_{ii}g_{jj} - g_{ij}^2) \le 0, \;\; IR_{g,2} \le 2g^2 g_1^2 g_{11} \sum_{i\neq 1} g_{ii} \le 0
\end{aligned}
$$

so that $g_{11} \le 0$ and $R_{g,2} \le 0$ at P_0. Let us define an operator $LX := X_t - \sum_{i,j=1}^n a_{ij} D_{ij} X$ for a_{ij} in (3.1.2). Then LX at $P_0 \in \Gamma_\varepsilon(t)$ satisfies

$$
\begin{aligned}
LX = &- \frac{2}{I^{3/2}} g^2 \sum_{i\neq 1} g_{i11}^2 - \frac{1}{JI^{3/2}} g^2 (2J - Ig) \sum_{i\neq 1} g_{i11}^2 - \frac{2}{I^{3/2}} Jg \sum_{i\neq 1} g_{i11}^2\\
&- \frac{2}{I^{3/2}} Jg \sum_{i,j\neq 1} g_{ij1}^2 - \frac{1}{\sqrt{I}} g^2 \sum_{i,j\neq 1} g_{ij1}^2 - \frac{1}{\sqrt{I}} Jg \sum_{i,j,k\neq 1} g_{ijk}^2 - \frac{1}{J^2 I^{3/2}} g^3 (2J - Ig) g_{111}^2\\
&- \frac{J - Ig}{J^2 I^{3/2}} 2g^2 (g_1 \sum_{i\neq 1} g_{ii} + 3g_1 g_{11}) g_{111} - \frac{J - Ig}{J^2 I^{3/2}} 4g^2 g_1 g_{11} \sum_{i\neq 1} g_{ii} g_{111}\\
&- \frac{1}{I^{3/2}} 2gg_1 \sum_{i\neq 1} g_{ii} \sum_{j\neq 1} g_{jj} g_{111} - \frac{2}{I^{3/2}} gg_1 \sum_{j\neq 1} g_{jj} g_{111}\\
&- \frac{1}{2I^{3/2}} g^3 \Big(gg_1 R_{g,2} + 4gg_1 g_{11} \sum_{j\neq 1} g_{jj} \Big) g_{111}\\
&+ \frac{1}{I^{5/2}} g \Big(-3g^2 g_1^3 \sum_{j\neq 1} g_{jj} + 3g^5 g_1^3 g_{11} \sum_{j\neq 1} g_{jj} \Big) g_{111}\\
&- \frac{2}{I^{3/2}} g^4 g_1 g_{11} \sum_{j\neq 1} g_{jj} g_{111} + \frac{2}{I^{3/2}} g^4 g_1 g_{11}^2 g_{111}\\
&+ \frac{3}{I^{5/2}} (g^3 g_1^2 \sum_{j\neq 1} g_{jj} - g^3 g_1^2 g_{11} + g_1^2) g^3 g_1 g_{11} g_{111}\\
&+ \frac{3}{I^{5/2}} (g^3 g_1^2 \sum_{j\neq 1} g_{jj} - g^3 g_1^2 g_{11} + g_1^2) g^2 g_1^3 g_{111} - \frac{3}{I^{3/2}} g^3 g_1^3 \sum_{j\neq 1} g_{jj} g_{111} + \frac{3}{I^{3/2}} g^3 g_1^3 g_{11} g_{111}\\
&- \frac{3}{I^{3/2}} (gg_{11} + g \sum_{j\neq 1} g_{jj}) g^3 g_1 g_{11} g_{111} - \frac{3}{I^{5/2}} (g_1^2 - g^3 g_1^2 g_{11}) g^3 g_1 g_{11} g_{111} - \frac{2}{I^{3/2}} g^4 g_1 g_{11}^2 g_{111}\\
&+ \frac{3}{I^{3/2}} g^4 g_1 g_{11} \sum_{j\neq 1} g_{jj} g_{111} + \frac{3}{I^{3/2}} g^4 g_1 g_{11}^2 g_{111}\\
&- \frac{3}{I^{3/2}} (gg_{11} + g \sum_{j\neq 1} g_{jj}) g^2 g_1^3 g_{111} - \frac{3}{I^{5/2}} (g_1^2 - g^3 g_1^2 g_{11}) g^2 g_1^3 g_{111} + \frac{1}{\sqrt{I}} gg_1 \sum_{j\neq 1} g_{jj} g_{111}\\
&+ \frac{1}{I^{3/2}} 2g^3 g_1^3 \sum_{j\neq 1} g_{jj} g_{111} - \frac{1}{I^{5/2}} 3g^4 g_1 g_{11} \sum_{j\neq 1} g_{jj} g_{111}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{I^{5/2}} 3g^3 g_1^3 \sum_{j\neq 1} g_{jj} g_{111} + \frac{1}{I^{3/2}} 2gg_1 \sum_{j\neq 1} g_{jj} g_{111} + \frac{1}{I^{3/2}} 2gg_1 \sum_{i\neq 1} g_{ii} \sum_{j\neq 1} g_{jj} g_{111} \\
&- \frac{1}{I^{3/2}} g^4 g_1 g_{11} \sum_{j\neq 1} g_{jj} g_{111} + \frac{1}{I^{3/2}} gg_1 \sum_{j\neq 1} g_{jj} g_{111} \\
&- \frac{1}{I^{3/2}} 2gg_1 \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii1} - \frac{1}{I^{3/2}} 4gg_1 g_{11} \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii1} \\
&- \frac{4}{\sqrt{I}} gg_1 \sum_{i\neq 1} g_{ii}^2 g_{ii1} + \frac{4}{\sqrt{I}} gg_1 \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii} g_{ii1} + \frac{4}{I^{3/2}} (g_1^2 + gg_{11}) g_1 \sum_{i\neq 1} g_{ii} g_{ii1} \\
&- \frac{3}{I^{3/2}} (gg_{11} + g \sum_{j\neq 1} g_{jj}) g^3 g_1 g_{11} \sum_{i\neq 1} g_{ii1} - \frac{3}{I^{5/2}} (g_1^2 - g^3 g_1^2 g_{11}) g^3 g_1 g_{11} \sum_{i\neq 1} g_{ii1} \\
&- \frac{2}{I^{3/2}} g^3 g_1 (gg_{11} + g_1^2) \sum_{i\neq 1} g_{ii} g_{ii1} - \frac{1}{\sqrt{I}} gg_1 \sum_{i\neq 1} g_{ii} g_{ii1} \\
&+ \frac{2}{I^{3/2}} (g^3 g_1 \sum_{j\neq 1} g_{jj} g_{11} + g_1 g_{11}) g \sum_{i\neq 1} g_{ii1} + \frac{3}{I^{3/2}} g^4 g_1 g_{11} \sum_{i\neq 1} g_{ii} g_{ii1} + \frac{3}{I^{3/2}} g^3 g_1^3 \sum_{i\neq 1} g_{ii} g_{ii1} \\
&- \frac{3}{I^{3/2}} (gg_{11} + g \sum_{j\neq 1} g_{jj}) g^2 g_1^3 \sum_{i\neq 1} g_{ii1} - \frac{3}{I^{5/2}} (g_1^2 - g^3 g_1^2 g_{11}) g^2 g_1^3 \sum_{i\neq 1} g_{ii1} \\
&+ \frac{1}{\sqrt{I}} (g_{11} + \sum_{j\neq 1} g_{jj}) gg_1 \sum_{i\neq 1} g_{ii1} + \frac{1}{I^{3/2}} (2g^2 g_1^2 \sum_{j\neq 1} g_{jj} - g^2 g_1^2 g_{11}) gg_1 \sum_{i\neq 1} g_{ii1} \\
&+ \frac{2}{I^{3/2}} gg_{11} \Big(g_1 \sum_{i\neq 1} g_{ii1} - g^2 gg_1 g_{11} \sum_{i\neq 1} g_{ii1} \Big) - \frac{1}{I^{5/2}} 3g^3 g_1^3 g_{11} \sum_{i\neq 1} g_{ii1} \\
&- \frac{1}{I^{3/2}} g^4 g_1 g_{11} (g_{11} \sum_{i\neq 1} g_{ii1} + \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii1} - \sum_{i\neq 1} g_{ii} g_{ii1}) + \frac{1}{I^{5/2}} 3g^6 g_1^3 g_{11}^2 \sum_{j\neq 1} g_{jj1} \\
&+ \frac{1}{I^{3/2}} gg_1 (g_{11} \sum_{i\neq 1} g_{ii1} + \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii1} - \sum_{i\neq 1} g_{ii} g_{ii1}) - \frac{3}{I^{5/2}} g^2 g_1^3 (gg_{11} + g_1^2) \sum_{i\neq 1} g_{ii1} \\
&+ \frac{1}{2I^{3/2}} \Big(4g_1 \sum_{j\neq 1} g_{jj} - g^3 g_1 R_{g,2} - 4g^3 g_1 g_{11} \sum_{j\neq 1} g_{jj} \Big) J \sum_{i\neq 1} g_{ii1} \\
&+ \frac{3}{I^{5/2}} \Big(- g^2 g_1^3 \sum_{j\neq 1} g_{jj} + g^5 g_1^3 g_{11} \sum_{j\neq 1} g_{jj} \Big) J \sum_{i\neq 1} g_{ii1} \\
&- \frac{4}{I^{3/2}} J g^3 g_1 \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii} g_{ii1} + \frac{4}{I^{3/2}} J g^3 g_1 \sum_{i\neq 1} g_{ii}^2 g_{ii1} - \frac{4}{I^{3/2}} J g_1 \sum_{i\neq 1} g_{ii} g_{ii1} \\
&+ \frac{1}{J^2 \sqrt{I}} g^2 (g_1 \sum_{i\neq 1} g_{ii} + 2g_1 g_{11} \sum_{i\neq 1} g_{ii} + 3g_1 g_{11})^2 - \frac{1}{\sqrt{I}} g \sum_{i\neq 1} g_{ii} \sum_{j\neq 1} g_{jj} \sum_{k\neq 1} g_{kk} \\
&- \frac{1}{I^{3/2}} g \sum_{j\neq 1} g_{jj} (2g_{11}^2 \sum_{i\neq 1} g_{ii} + 3g_{11}^2 + g_{11} \sum_{i\neq 1} g_{ii})
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\sqrt{I}} g \sum_{i\neq 1} g_{ii}(2g_{ii}^3 - 2g_{ii}^2 g_{11} - 2g_{ii}^2 \sum_{j\neq 1} g_{jj} - 2g_{ii}^2 - g_{ii}g_{11} - g_{ii}\sum_{j\neq 1} g_{jj}) \\
&+ \frac{1}{\sqrt{I}} g \sum_{j\neq 1} g_{jj} \sum_{i\neq 1}(2g_{ii}^3 - 2g_{ii}^2 g_{11} - 2g_{ii}^2 \sum_{j\neq 1} g_{jj} - 2g_{ii}^2 - g_{ii}g_{11}) \\
&+ \frac{1}{I^{3/2}}(g_1^2 + gg_{11}) \sum_{i\neq 1}(2g_{ii}^3 - 2g_{ii}^2 g_{11} - 2g_{ii}^2 \sum_{j\neq 1} g_{jj} - 2g_{ii}^2 - g_{ii}g_{11} - g_{ii}\sum_{j\neq 1} g_{jj}) \\
&+ \frac{1}{2I^{3/2}} gg_{11}R_{g,2} - \frac{3}{I^{5/2}} g^2 g_1^2 (gg_{11} + g_1^2) g_{11} \sum_{j\neq 1} g_{jj} \\
&+ \frac{1}{2I^{3/2}} gg_{11}\Big(4g_{11}\sum_{j\neq 1} g_{jj} - g^3 R_{g,2} g_{11} - 4g^3 g_{11}^2 \sum_{j\neq 1} g_{jj}\Big) \\
&+ \frac{1}{I^{5/2}} gg_{11}\Big(-g^5 g_1^2 g_{11} R_{g,2} + 5g^5 g_1^2 g_{11}^2 \sum_{j\neq 1} g_{jj} - 5g^2 g_1^2 g_{11}\sum_{j\neq 1} g_{jj}\Big) \\
&- \frac{5}{I^{7/2}}\Big(4g_1 \sum_{j\neq 1} g_{jj} - g^2(gg_1 R_{g,2} + 4gg_1 g_{11}\sum_{j\neq 1} g_{jj})\Big) g^3 g_1 g_{11}^2 \\
&- \frac{15}{I^{7/2}}\Big(-g^2 g_1^3 \sum_{j\neq 1} g_{jj} + g^5 g_1^3 g_{11}\sum_{j\neq 1} g_{jj}\Big) g^3 g_1 g_{11}^2 \\
&- \frac{3}{2I^{3/2}} g^3 g_1 R_{g,2} - \frac{6}{I^{3/2}} g^3 g_1 g_{11} \sum_{j\neq 1} g_{jj} \\
&+ \frac{1}{I^{5/2}} gg_1 g_{11}\Big(-2gg_1^3 \sum_{j\neq 1} g_{jj} - g^4 g_1^3 R_{g,2} + 11 g^4 g_1^3 g_{11}\sum_{j\neq 1} g_{jj}\Big) \\
&- \frac{5}{2I^{5/2}} g^2 g_1^3 g_{11}\Big(4g_1 \sum_{j\neq 1} g_{jj} - g^2(gg_1 R_{g,2} + 4gg_1 g_{11}\sum_{j\neq 1} g_{jj})\Big) \\
&+ \frac{15}{I^{7/2}} g^2 g_1^3 g_{11}\Big(g^2 g_1^3 \sum_{j\neq 1} g_{jj} - g^5 g_1^3 g_{11}\sum_{j\neq 1} g_{jj}\Big) \\
&+ \frac{1}{I^{5/2}} gg_1\Big(-6g^2 g_1 g_{11}\sum_{j\neq 1} g_{jj}g_{11} + g^2 g_1 g_{11} R_{g,2} - 12 gg_1^2 g_1 \sum_{j\neq 1} g_{jj} g_{11}\Big) \\
&- \frac{5}{2I^{5/2}} g^2 g_1^2 (gg_{11} + g_1^2) R_{g,2} + \frac{15}{2I^{7/2}} g^3 g_1^4 (gg_{11} + g_1^2)^2 \sum_{j\neq 1} g_{jj} \\
&+ \frac{1}{I^{5/2}} gg_1^2(-6gg_1^2 g_{11}\sum_{j\neq 1} g_{jj} + gg_1^2 R_{g,2} - 3g_1^4 \sum_{j\neq 1} g_{jj}) \\
&+ \frac{1}{2I^{3/2}} JR_{g,2}\sum_{i\neq 1} g_{ii} - \frac{3}{I^{5/2}} Jgg_1^2(gg_{11} + g_1^2)\sum_{i\neq 1} g_{ii}\sum_{j\neq 1} g_{jj} \\
&+ \frac{1}{I^{3/2}} 2Jg_{11}\sum_{i\neq 1} g_{ii}^2 + \frac{1}{I^{3/2}} 2J \sum_{j\neq 1} g_{jj}\sum_{i\neq 1} g_{ii}^2 - \frac{1}{I^{3/2}} 2J \sum_{i\neq 1} g_{ii}^3
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2I^{3/2}} Jg^3 R_{g,2} \sum_{i\neq 1} g_{ii}^2 - \frac{1}{I^{3/2}} 2Jg^3 g_{11} \sum_{i\neq 1} g_{ii}^3 - \frac{1}{I^{3/2}} 2Jg^3 \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii}^3 \\
&+ \frac{1}{I^{3/2}} 2Jg^3 \sum_{i\neq 1} g_{ii}^4 + \frac{1}{I^{5/2}} 3Jg^5 g_1^2 g_{11} \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii}^2 - \frac{1}{I^{5/2}} 3Jg^2 g_1^2 \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii}^2 \\
&+ \frac{1}{I^{3/2}} g_1 g_{11} (1 + \sum_{i\neq 1} g_{ii}) \Big(- g^3 g_1 R_{g,2} + (4g_1 + 2g^3 g_1 g_{11}) \sum_{j\neq 1} g_{jj} \Big) \\
&- \frac{6}{I^{5/2}} g g_1^2 (g g_{11} + g_1^2)^2 (\sum_{j\neq 1} g_{jj} + \sum_{i\neq 1} g_{ii} \sum_{j\neq 1} g_{jj}) + \frac{1}{I^{3/2}} g_1^2 (1 + \sum_{i\neq 1} g_{ii}) R_{g,2} \\
&+ \frac{1}{2\sqrt{I}} (g_{11} + \sum_{i\neq 1} g_{ii}) \Big(g R_{g,2} + 2g_1^2 \sum_{j\neq 1} g_{jj} \Big) - \frac{1}{I^{3/2}} g^2 g_1^2 (g_1^2 + g g_{11}) (\sum_{j\neq 1} g_{jj} + \sum_{i\neq 1} g_{ii} \sum_{j\neq 1} g_{jj}) \\
= &- \Big(31J - 20gI\Big) \frac{1}{20J^2 I^{3/2}} g^3 g_{111}^2 - \frac{1}{JI^{3/2}} \Big(2gJ^2 + g^2(4J - Ig) \Big) \sum_{i\neq 1} g_{i11}^2 \\
&- \frac{1}{J\sqrt{I}} g \sum_{i,j,k\neq 1} (Jg_{ijk})^2 - \frac{1}{J^2\sqrt{I}} g^2 \sum_{i,j\neq 1} (Jg_{ij1})^2 - \frac{2}{I^{3/2}} Jg \sum_{i,j\neq 1, i\neq j} g_{ij1}^2 - \frac{1}{4I^{3/2}} Jg \sum_{i\neq 1} g_{ii1}^2 \\
&- \frac{1}{20JI^{3/2}} g \Big(\Big(g g_{111} - \frac{20J}{I} g_1^3 g \sum_{j\neq 1} g_{jj} \Big)^2 + (g g_{111} + 5Jg_1 g^2 R_{g,2})^2 + (g g_{111} - 10Jg_1 g g_{11} g \sum_{j\neq 1} g_{jj})^2 \\
&+ \Big(g g_{111} + \frac{90J}{I} g_1 (g g_{11} + g_1^2) g \sum_{j\neq 1} g_{jj} \Big)^2 + \Big(g g_{111} - \frac{20I}{J} g_1 g \sum_{i\neq 1} g_{ii} \Big)^2 + \Big(g g_{111} - \frac{60I}{J} g_1 g g_{11} \Big)^2 \\
&+ (g g_{111} + 20 g_1 \sum_{i\neq 1} g_{ii})^2 + (g g_{111} + 60 g_1 g_{11})^2 + \Big(g g_{111} - \frac{40I}{J} g_1 g g_{11} \sum_{i\neq 1} g_{ii} \Big)^2 \Big) \\
&- \frac{1}{8I^{3/2}} Jg \sum_{i\neq 1} \Big(\Big(g_{ii1} + \frac{8}{J} g g_1 g^2 g_{11} \sum_{j\neq 1} g_{jj} \Big)^2 + \Big(g_{ii1} - \frac{24}{J} g_1 g_{11} \Big)^2 + \Big(g_{ii1} + \frac{24}{JI} g g_1 f_{11}^2 \Big)^2 \\
&+ (g_{ii1} + 2g_1 g^2 R_{g,2})^2 + \Big(g_{ii1} + \frac{4}{I} g_1^3 g \sum_{j\neq 1} g_{jj} \Big)^2 + \Big(g_{ii1} - \frac{4}{I} g^2 g_1^3 g^2 g_{11} \sum_{j\neq 1} g_{jj} \Big)^2 + \Big(g_{ii1} + \frac{16}{J} g_1 g_{ii}^2 \Big)^2 \\
&+ \Big(g_{ii1} + \frac{8}{I} g_1 g^2 g_{11} \sum_{j\neq 1} g_{jj} \Big)^2 + \Big(g_{ii1} - \frac{16I}{J} g_1 \sum_{j\neq 1} g_{jj} g_{ii} \Big)^2 + \Big(g_{ii1} - \frac{16}{J} g_1 g_{11} g_{ii} \Big)^2 \\
&+ \Big(g_{ii1} + \frac{24}{J} g_1 g_{ii} \Big)^2 + \Big(g_{ii1} - \frac{8}{J} g g_1 g^2 g_{11} g_{ii} \Big)^2 + (g_{ii1} + 16 g_1 g^2 \sum_{j\neq 1} g_{jj} g_{ii})^2 + \Big(g_{ii1} - \frac{16}{J} g g_1 g^2 g_{ii}^2 \Big)^2 \Big) \\
&+ \Bigg(\frac{5J}{4I^{7/2}} g g_1^2 \Big(16(g \sum_{j\neq 1} g_{jj})^2 + I^2 (g^2 R_{g,2})^2 + 4I^2 (g^2 g_{11} \sum_{j\neq 1} g_{jj})^2 \Big) \\
&+ \frac{1}{20J^3 I^{7/2}} g \Big(400 (I^2 g_1 g \sum_{i\neq 1} g_{ii})^2 + (90J^2 g_1 f_{11} g \sum_{j\neq 1} g_{jj})^2 + (60I^2 g_1)^2 (g g_{11})^2 \Big) \\
&+ \frac{n-1}{8I^{3/2}} Jg \Big(2 \Big(\frac{24}{JI} g g_1 f_{11}^2 \Big)^2 + \Big(\frac{8}{J} g g_1 g^2 g_{11} \sum_{j\neq 1} g_{jj} \Big)^2 + (2 g_1 g^2 R_{g,2})^2 + \Big(\frac{4}{I} g_1^3 g \sum_{j\neq 1} g_{jj} \Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{4}{I}g^2g_1^3g^2g_{11}\sum_{j\neq 1}g_{jj}\Big)^2+\Big(\frac{8}{I}g_1g^2g_{11}\sum_{j\neq 1}g_{jj}\Big)^2\Big)+\frac{g^2g_1}{J^2\sqrt{I}}\Big((\sum_{i\neq 1}g_{ii})^2+9g_{11}^2\Big)\\
&+\frac{1}{8I^{3/2}}Jg\Big(\Big(\frac{8}{J}gg_1g^2g_{11}\Big)^2\sum_{i\neq 1}g_{ii}^2+(16g_1g^2\sum_{j\neq 1}g_{jj})^2\sum_{i\neq 1}g_{ii}^2+\Big(\frac{16}{J}gg_1g^2\Big)^2\sum_{i\neq 1}g_{ii}^4\Big)\\
&+\frac{1}{2I^{3/2}}g\Big(-g^2g_{11}^2(gR_{g,2})-4g^3g_{11}^3\sum_{j\neq 1}g_{jj}-3gg_1(gR_{g,2})-12gg_1(gg_{11})\sum_{j\neq 1}g_{jj}\\
&-2g_1^2(gg_{11})(gR_{g,2})+4g_1^2(gg_{11})^2\sum_{i\neq 1}g_{ii}\Big)\\
&+\frac{1}{I^{5/2}}gg_{11}\Big(-3gg_1^2f_{11}\sum_{j\neq 1}g_{jj}-g^3g_1^2(gg_{11})(gR_{g,2})+5g^3g_1^2(gg_{11})^2\sum_{j\neq 1}g_{jj}-11gg_1^2(gg_{11})\sum_{j\neq 1}g_{jj}\\
&-30gg_1^4\sum_{j\neq 1}g_{jj}+\frac{3}{2}g^3g_1^4(gR_{g,2})+21g^3g_1^4(gg_{11})\sum_{j\neq 1}g_{jj}+gg_1^2(gR_{g,2})+3Jg^4g_1^2\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2\Big)\\
&+\frac{1}{2I^{5/2}}\Big(-5g_1^2f_{11}(g^2R_{g,2})+2g_1^4(g^2R_{g,2})-6g_1^6g\sum_{j\neq 1}g_{jj}\Big)\\
&+\frac{5}{2I^{7/2}}g^3g_1^2\Big(8(gg_{11})^3\sum_{j\neq 1}g_{jj}+6g_1^2f_{11}(gg_{11})\sum_{j\neq 1}g_{jj}-6g^2g_1^2f_{11}(gg_{11})^2\sum_{j\neq 1}g_{jj}+3g_1^2f_{11}^2\sum_{j\neq 1}g_{jj}\Big)\Big)\\
&+\Bigg(\frac{6}{J^2\sqrt{I}}g_1(gg_{11})(g\sum_{i\neq 1}g_{ii})-\frac{1}{I^{3/2}}g_1^2f_{11}(g^2\sum_{i\neq 1}g_{ii}+(g\sum_{i\neq 1}g_{ii})^2)-\frac{6}{I^{5/2}}g_1^2f_{11}^2g\sum_{i\neq 1}g_{ii}\\
&+\frac{5}{I^{7/2}}\Big(-4g_1^2(gg_{11})^2(g\sum_{j\neq 1}g_{jj})+g^2g_1^2(gg_{11})^2g^2R_{g,2}\Big)\Bigg)\\
&+\Bigg(\frac{4}{J^2\sqrt{I}}g_1g_{11}(g\sum_{i\neq 1}g_{ii})^2\\
&+\Big(-Ig_1^2(g^3g_{11}R_{g,2})-2IJ(g^3\sum_{j\neq 1}g_{jj}^3)-(3J+6)g_1^2f_{11}(g\sum_{j\neq 1}g_{jj})-3Jg_1^2(g^2\sum_{j\neq 1}g_{jj}^2)\Big)\frac{1}{I^{5/2}}\sum_{i\neq 1}g_{ii}\Bigg)\\
&+\Bigg(\frac{1}{JI^{3/2}}\Big(g_1^2(gg_{11})\Big(180+72(n-1)\Big)-2J^2(g^3\sum_{i\neq 1}g_{ii}^3)\Big)g_{11}-\frac{3}{I^{3/2}}(g\sum_{j\neq 1}g_{jj})g_{11}^2\Bigg)\\
&+\Bigg(\frac{20}{J^3I^{3/2}}g_1^2(g\sum_{j\neq 1}g_{jj})\Big(J^2+4I^2(gg_{11})^2\Big)\sum_{i\neq 1}g_{ii}+\frac{1}{JI^{3/2}}72g_1^2(g\sum_{i\neq 1}g_{ii}^2)\\
&+\frac{12}{J^2\sqrt{I}}g_1g^2g_{11}^2\sum_{i\neq 1}g_{ii}+\frac{2}{I^{3/2}}g_1^2g^3g_{11}^2(\sum_{i\neq 1}g_{ii})^2+\frac{1}{2I^{3/2}}J(-g^2R_{g,2})(g\sum_{i\neq 1}g_{ii}^2)+\frac{1}{I^{3/2}}2J(g^3\sum_{i\neq 1}g_{ii}^4)\\
&-\frac{1}{I^{3/2}}g_1^2(4+I)(-g_{11})\sum_{i\neq 1}g_{ii}-\frac{4}{JI^{3/2}}g_1^2(-gg_{11})(\sum_{i\neq 1}g_{ii})^2-\frac{1}{I^{3/2}}f_{11}\Big(2\sum_{i\neq 1}g_{ii}^2+(\sum_{i\neq 1}g_{ii})^2\Big)\\
&-\frac{1}{\sqrt{I}}(-gg_{11})\sum_{i\neq 1}g_{ii}^2-\frac{2}{I^{5/2}}g_1^2(\sum_{i\neq 1}g_{ii})^2-\frac{1}{\sqrt{I}}(g\sum_{j\neq 1}g_{jj})\Big((\sum_{i\neq 1}g_{ii})^2+2\sum_{i\neq 1}g_{ii}^2\Big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\bigg(\Big(-\frac{6}{I^{5/2}}g_1^2g_{11}\sum_{i\neq 1}g_{ii}+\frac{12}{JI^{3/2}}gg_1^2g_{11}^2\sum_{i\neq 1}g_{ii}-\frac{1}{I^{3/2}}(g_1^2+gg_{11})g_{11}\sum_{i\neq 1}g_{ii}+\frac{2}{I^{3/2}}gg_{11}^2\sum_{j\neq 1}g_{jj}\\
&\quad+\frac{4}{J^2\sqrt{I}}g_1g^2g_{11}^2(\sum_{i\neq 1}g_{ii})^2-\frac{1}{I^{3/2}}gg_{11}(\sum_{i\neq 1}g_{ii})^2+\frac{2}{\sqrt{I}}g\sum_{i\neq 1}g_{ii}^3+\frac{1}{\sqrt{I}}g\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2\\
&\quad-\frac{1}{\sqrt{I}}gg_{11}(\sum_{i\neq 1}g_{ii})^2+\frac{1}{\sqrt{I}}g_1^2(\sum_{i\neq 1}g_{ii})^2\Big)\\
&\quad+\Big(\frac{2}{I^{3/2}}Jg_{11}\sum_{i\neq 1}g_{ii}^2+\frac{4}{I^{3/2}}g_1^2g_{11}(\sum_{i\neq 1}g_{ii})^2+\frac{2}{\sqrt{I}}gg_{11}\sum_{i\neq 1}g_{ii}^3\\
&\quad-\frac{2}{\sqrt{I}}g\sum_{i\neq 1}g_{ii}^4-\frac{2}{\sqrt{I}}g\sum_{i\neq 1}g_{ii}^2(\sum_{j\neq 1}g_{jj})^2-\frac{2}{I^{3/2}}(g_1^2+gg_{11})\sum_{i\neq 1}g_{ii}^2\sum_{j\neq 1}g_{jj}-\frac{2}{I^{3/2}}J\sum_{i\neq 1}g_{ii}^3\Big)\bigg)\\
&\quad+\bigg(\Big(\frac{32}{JI^{3/2}}g_1^2gg_{11}^2\sum_{i\neq 1}g_{ii}^2+\frac{1}{JI^{3/2}}32I^2g_1^2g(\sum_{i\neq 1}g_{ii})^2\sum_{j\neq 1}g_{jj}^2+\frac{32}{JI^{3/2}}g_1^2g\sum_{i\neq 1}g_{ii}^4\\
&\quad-\frac{4}{I^{5/2}}g_1^2g_{11}(\sum_{i\neq 1}g_{ii})^2+\frac{8}{JI^{3/2}}gg_1^2(g_{11}\sum_{i\neq 1}g_{ii})^2-\frac{2}{I^{3/2}}(g_1^2+gg_{11})g_{11}\sum_{i\neq 1}g_{ii}^2\\
&\quad+\frac{2}{\sqrt{I}}g\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^3+\frac{2}{\sqrt{I}}g\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^3-\frac{2}{\sqrt{I}}gg_{11}\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2\\
&\quad+\frac{2}{I^{3/2}}(g_1^2+gg_{11})\sum_{i\neq 1}g_{ii}^3+\frac{2}{I^{3/2}}J\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2\Big)-\frac{2}{I^{3/2}}gg_{11}^2(\sum_{j\neq 1}g_{ii})^3\bigg)\\
&\quad+\frac{1}{2I^{3/2}}\Big((J+2g_1^2)\sum_{i\neq 1}g_{ii}+2g_1^2+I(g\sum_{i\neq 1}g_{ii})+(gg_{11})+I(gg_{11})\Big)R_{g,2}
\end{aligned}
$$

As ε tends to zero, LX at $P_0\in\Gamma_\varepsilon(t)$ becomes

$$
\begin{aligned}
LX &:= X_t-\sum_{i,j=1}^{n}a_{ij}D_{ij}X=-\Big(31J-20gI\Big)\frac{1}{20J^2I^{3/2}}g^3g_{111}^2-\frac{1}{JI^{3/2}}\Big(2gJ^2+g^2(4J-Ig)\Big)\sum_{i\neq 1}g_{i11}^2\\
&-\frac{1}{J\sqrt{I}}g\sum_{i,j,k\neq 1}(Jg_{ijk})^2-\frac{1}{J^2\sqrt{I}}g^2\sum_{i,j\neq 1}(Jg_{ij1})^2-\frac{2}{I^{3/2}}Jg\sum_{i,j\neq 1,i\neq j}g_{ij1}^2-\frac{1}{4I^{3/2}}Jg\sum_{i\neq 1}g_{ii1}^2\\
&-\frac{1}{20JI^{3/2}}g\Big(\Big(gg_{111}-\frac{20J}{I}g_1^3g\sum_{j\neq 1}g_{jj}\Big)^2+(gg_{111}+5Jg_1g^2R_{g,2})^2+(gg_{111}-10Jg_1gg_{11}g\sum_{j\neq 1}g_{jj})^2\\
&+\Big(gg_{111}+\frac{90J}{I}g_1(gg_{11}+g_1^2)g\sum_{j\neq 1}g_{jj}\Big)^2+\Big(gg_{111}-\frac{20I}{J}g_1g\sum_{i\neq 1}g_{ii}\Big)^2+\Big(gg_{111}-\frac{60I}{J}g_1gg_{11}\Big)^2\\
&+(gg_{111}+20g_1\sum_{i\neq 1}g_{ii})^2+(gg_{111}+60g_1g_{11})^2+\Big(gg_{111}-\frac{40I}{J}g_1gg_{11}\sum_{i\neq 1}g_{ii}\Big)^2\Big)\\
&-\frac{1}{8I^{3/2}}Jg\sum_{i\neq 1}\Big(\Big(g_{ii1}+\frac{8}{J}gg_1g^2g_{11}\sum_{j\neq 1}g_{jj}\Big)^2+\Big(g_{ii1}-\frac{24}{J}g_1g_{11}\Big)^2+\Big(g_{ii1}+\frac{24}{JI}gg_1f_{11}^2\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ (g_{ii1} + 2g_1 g^2 R_{g,2})^2 + \Big(g_{ii1} + \frac{4}{I} g_1^3 g \sum_{j\neq1} g_{jj}\Big)^2 + \Big(g_{ii1} - \frac{4}{I} g^2 g_1^3 g^2 g_{11} \sum_{j\neq1} g_{jj}\Big)^2 + \Big(g_{ii1} + \frac{16}{J} g_1 g_{ii}^2\Big)^2 \\
&+ \Big(g_{ii1} + \frac{8}{I} g_1 g^2 g_{11} \sum_{j\neq1} g_{jj}\Big)^2 + \Big(g_{ii1} - \frac{16I}{J} g_1 \sum_{j\neq1} g_{jj} g_{ii}\Big)^2 + \Big(g_{ii1} - \frac{16}{J} g_1 g_{11} g_{ii}\Big)^2 \\
&+ \Big(g_{ii1} + \frac{24}{J} g_1 g_{ii}\Big)^2 + \Big(g_{ii1} - \frac{8}{J} g g_1 g^2 g_{11} g_{ii}\Big)^2 + (g_{ii1} + 16 g_1 g^2 \sum_{j\neq1} g_{jj} g_{ii})^2 + \Big(g_{ii1} - \frac{16}{J} g g_1 g^2 g_{ii}^2\Big)^2\Big) \\
&+ \Big(\frac{1}{I^{5/2}} O(1) + \frac{n-1}{8I^{3/2}} J O(g) + \frac{1}{8I^{3/2}} O(g) + \frac{5}{2I^{7/2}} O(g^2)\Big) \\
&+ \frac{1}{2I^{3/2}} \Big((J + 2g_1^2) \sum_{i\neq1} g_{ii} + 2g_1^2 + I(g \sum_{i\neq1} g_{ii}) + (g g_{11}) + I(g g_{11})\Big) R_{g,2} \\
&+ \Big(-\frac{3}{I^{3/2}} (g \sum_{j\neq1} g_{jj}) g_{11}^2 + \frac{4}{J^2\sqrt{I}} g_1 (g\sum_{i\neq1} g_{ii})^2 g_{11} + \frac{1}{JI^{3/2}} \Big(g_1^2 (g g_{11})\big(180 + 72(n-1)\big) - 2J^2 (g^3 \sum_{i\neq1} g_{ii}^3)\Big) g_{11}\Big) \\
&+ \Big(-\frac{2}{I^{5/2}} g_1^2 (\sum_{i\neq1} g_{ii})^2 - \frac{1}{I^{3/2}} f_{11} \big(2\sum_{i\neq1} g_{ii}^2 + (\sum_{i\neq1} g_{ii})^2\big) - \frac{1}{\sqrt{I}} (g \sum_{j\neq1} g_{jj}) \big((\sum_{i\neq1} g_{ii})^2 + 2\sum_{i\neq1} g_{ii}^2\big) \\
&+ \frac{2}{I^{3/2}} O(1) \sum_{i\neq1} g_{ii} + \frac{1}{2I^{3/2}} O(1)(g \sum_{i\neq1} g_{ii}^2) + \frac{1}{I^{3/2}} O(1)(g^3 \sum_{i\neq1} g_{ii}^4) \\
&- \frac{1}{I^{3/2}} g_1^2 (4+I)(-g_{11}) \sum_{i\neq1} g_{ii} - \frac{4}{JI^{3/2}} g_1^2 (-g g_{11}) (\sum_{i\neq1} g_{ii})^2 - \frac{1}{\sqrt{I}} (-g g_{11}) \sum_{i\neq1} g_{ii}^2 \Big) \\
&+ \Big(\Big(\frac{2}{I^{3/2}}(-J + I g g_{11}) \sum_{i\neq1} g_{ii}^3 - \frac{2}{\sqrt{I}} g \sum_{i\neq1} g_{ii}^4 - \frac{2}{I^{3/2}}(Ig\sum_{k\neq1} g_{kk} + f_{11}) \sum_{i\neq1} g_{ii}^2 \sum_{j\neq1} g_{jj} + \frac{2}{\sqrt{I}} g \sum_{i\neq1} g_{ii}^3\Big) \\
&+ \frac{1}{I^{3/2}} O(1)(\sum_{i\neq1} g_{ii})^2 + \frac{1}{\sqrt{I}} O(1) \sum_{i\neq1} g_{ii}^2 + \Big(\frac{2}{I^{3/2}} J g_{11} \sum_{i\neq1} g_{ii}^2 + \frac{4}{I^{3/2}} g_1^2 g_{11} (\sum_{i\neq1} g_{ii})^2 + \frac{2}{I^{3/2}} O(1) g_{11} \sum_{j\neq1} g_{jj}\Big)\Big) \\
&+ \Big(-\frac{2}{I^{3/2}} g g_{11}^2 (\sum_{j\neq1} g_{ii})^3 + \Big(\frac{32}{JI^{3/2}} g_1^2 g g_{11}^2 \sum_{i\neq1} g_{ii}^2 - \frac{4}{I^{5/2}} g_1^2 g_{11} (\sum_{i\neq1} g_{ii})^2\Big) \\
&+ \Big(\frac{1}{JI^{3/2}} O(1) \sum_{i\neq1} g_{ii} \sum_{j\neq1} g_{jj}^2 + \frac{32}{JI^{3/2}} g_1^2 g \sum_{i\neq1} g_{ii}^4 + \frac{4}{\sqrt{I}} O(1) \sum_{i\neq1} g_{ii}^3\Big)\Big) \le C
\end{aligned}
$$

for all $g < \varepsilon_0'$, for some constants $C > 0,\ 0 < \varepsilon_0' < \varepsilon_0$, hence X satisties $X_t \le LX \le C$ at its interior maximum point $P_0(t)$ at each time $0 \le t \le T$, so X is bounded above on level sets Γ_ε for all $\varepsilon < \varepsilon_0'$. Consequently, X is bounded above. □

Corollary 4.2.2. *Let us assume the conditions in the subsection 1.4. Then there exists a constant $c > 0$ satisfying*

$$
c \le \sum_{\tau:\ \textit{tangential}} g_{\tau\tau} \le c^{-1}. \tag{4.2.18}
$$

for $0 \le g \le 1$ and $0 \le t \le T$.

Proof. From the inequality (3.2.49), it holds that for some constant $c > 0$

$$\sum_{i,j}(f_{ii}f_{jj} - f_{ij}^2) = 2\sum_{i\neq 1}(f_{11}f_{ii} - f_{i1}^2) + \sum_{i,j\neq 1}(f_{ii}f_{jj} - f_{ij}^2) \geq cg \tag{4.2.19}$$

where $e_1 = \nu$ is the unit normal to the level set, which yields

$$\Big(\sum_{i\neq 1} f_{ii}\Big)^2 + 2f_{11}\sum_{i\neq 1} f_{ii} = (\sum_{i\neq 1} f_{ii} + 2f_{11})g\sum_{i\neq 1} g_{ii} \geq cg. \tag{4.2.20}$$

As $\Delta f \geq 0$ is bounded above, there exists some constant $\widetilde{v} > 0$ such that $\sum_{i\neq 1} g_{ii} \geq \widetilde{c}$, which implies the inequality (4.2.18). □

We can also examine the bounds for the second-order derivatives of f.

Corollary 4.2.3. *Let us assume the conditions in the subsection 1.4. Then there exists a constant* $c > 0$ *satisfying*

$$c \leq f_{\nu\nu},\ \frac{1}{g}\sum_{\tau:\ tangential} f_{\tau\tau} \leq c^{-1} \ \ and\ \ \sqrt{\frac{1}{g}\sum_{\tau:\ tangential} f_{\tau\nu}^2} \leq c^{-1}. \tag{4.2.21}$$

where ν *is the unit normal to the level set, for* $0 \leq g \leq 1$ *and* $0 \leq t \leq T$.

Proof. Let us use a coordinate system where $e_1 = \nu$ is the unit normal to the level set of g. From the inequality (4.2.18), we see that $c \leq g^{-1}\sum_{i\neq 1} f_{ii} \leq c^{-1}$.

With the convexity of f and the boundedness $0 \leq f_{11} \leq c^{-1}$, and the inequality (4.2.20) implies $\sum_{i\neq 1} f_{i1}^2 \leq \sum_{i\neq 1} f_{11}f_{ii} \leq c^{-1}g$. Since $cg^2 \leq \left(\sum_{i\neq 1} f_{ii}\right)^2 \leq c^{-1}g^2$ because of the inequality (4.2.18), we see that $f_{11} \geq c$ from the inequality (4.2.20). □

Consequently, the following result holds as well.

Lemma 4.2.4. *Let* $P_0 = (x_0, t_0) \in \Gamma(t_0),\ 0 \leq t_0 \leq T$, *with its position vector* $\mathbf{P_0}$ *and* $n_0 = \frac{\mathbf{P_0}}{|\mathbf{P_0}|}$. *Then there exist positive constants* C *and* η *depending only on initial data and the constant* ρ_0, *satisfying*

$$C \leq g_{n_0} \leq C^{-1},\ C \leq f_{n_0 n_0} \leq C^{-1} \tag{4.2.22}$$

for all $P = (x, t)$ *where* $f(x, t) > 0$, $|P - P_0| \leq \eta,\ 0 \leq t \leq t_0$.

Proof. Let $\tau(P)$ be the unit vector in the direction of the tangential projection of n_0 and θ be the angle between n_0 and the outward normal $\nu(P)$, for $P = (x, t)$ where $x \in \Omega(t)$. Then we get

$$g_{n_0} = \sin\theta g_\tau + \cos\theta g_\nu,\ f_{n_0 n_0} = \cos^2\theta f_{\nu\nu} + 2\cos\theta\sin\theta f_{\nu\tau} + \sin^2\theta f_{\tau\tau}.$$

By Lemma 3.2.2, there exists $\eta > 0$ such that $\cos\theta = n_0 \cdot \nu \geq \gamma > 0$ for any $P = (x, t) \in \Omega(t)$ with $|P - P_0| \leq \eta,\ 0 \leq t \leq t_0$. Hence, the desired result follows by Corollary 4.2.3. □

Lemma 4.2.5. *Let us assume the conditions in the subsection 1.4. Then there exists a uniform constant $C > 0$ satisfying*

$$R_{g,2} = \sum_{i,j=1}^{n}(g_{ii}g_{jj} - g_{ij}^2) \geq -C \tag{4.2.23}$$

for $0 \leq g \leq 1$ and $0 \leq t \leq T$.

Proof. We may assume $0 \leq g < \varepsilon$, for sufficiently small $0 < \varepsilon \leq \varepsilon_0$, since $|R_{g,2}|$ is bounded above on $\{\frac{\varepsilon}{2} \leq g \leq 1\}$ by the existence of a strictly smooth solution from [11]. To analyze $R_{g,2} = \sum_{i,j\neq\nu}(g_{ii}g_{jj} - g_{ij}^2) + 2\sum_{i\neq\nu}(g_{\nu\nu}g_{ii} - g_{\nu i}^2)$, we define a quantity

$$X := \frac{2\sum_{i\neq\nu}(g_{\nu\nu}g_{ii} - g_{\nu i}^2)}{\sum_{i,j}(g_i^2 g_{jj} + g_j^2 g_{ii} - 2g_i g_j g_{ij})} + \exp\left(b|\nabla g|^2\right) \tag{4.2.24}$$

on $\Omega(t)$, $0 \leq t \leq T$, where $\nu \in \mathbb{R}^n$ is the unit normal to the level set of g and we always take $e_1 = \nu$ for simplicity. Note that X is invariant under coordinate rotation. Getting a uniform lower bound of X, we obtain a uniform lower bound of $R_{g,2}$. We will use the maximum principle to show that $X \geq 0$ for all time, for some constant $b > 0$. First, we take $b > 0$ sufficiently large such that $X > 0$ on Ω at time $t = 0$ and $X > 1$ on the level set $\{g = 1\}$ for all time, and we will determine the value of the constant b later.

Suppose that X gets minimum at an interior point $P_0 = P_0(t)$ of $\Omega(t)$. At P_0, we choose a coordinate system where $e_1 = \nu$ and $g_{ij} = 0$ for $i \neq j$. Then

$$X = \frac{g_{\nu\nu}}{|\nabla g|^2} + \exp\left(b|\nabla g|^2\right) = \frac{g_{11}}{g_1^2} + \exp\left(bg_1^2\right). \tag{4.2.25}$$

The quantity X at P_0 evolves in time as

$$X_t = \frac{1}{g_1^2}g_{11t} + \left(2bg_1\exp\left(bg_1^2\right) - \frac{2g_{11}}{g_1^3}\right)g_{1t} \tag{4.2.26}$$

with derivatives

$$\begin{aligned}
X_1 =& \frac{1}{g_1^2}g_{111} + \left(2bg_1\exp\left(bg_1^2\right) - \frac{2g_{11}}{g_1^3}\right)g_{11},\ X_m = \frac{1}{g_1^2}g_{11m} \text{ for } m \neq 1\\
X_{11} =& \frac{1}{g_1^2}g_{1111} + \left(2bg_1\exp\left(bg_1^2\right) - \frac{6g_{11}}{g_1^3}\right)\left(-2bg_1^3\exp\left(bg_1^2\right) + \frac{2g_{11}}{g_1}\right)g_{11}\\
&+ \left((4b^2g_1^2 + 2b)\exp\left(bg_1^2\right) + \frac{6g_{11}}{g_1^4}\right)g_{11}^2\\
X_{mm} =& \frac{1}{g_1^2}g_{mm11} - \frac{2}{g_1^2\sum_{k\neq1}g_{kk}}\sum_{i\neq1}g_{im1}^2 + 2bg_1g_{mm1}\exp\left(bg_1^2\right) - \frac{2g_{11}}{g_1^3}g_{mm1} + \frac{4g_{11}}{g_1^3\sum_{k\neq1}g_{kk}}g_{mm}g_{mm1}\\
&+ 2bg_{mm}^2\exp\left(bg_1^2\right) - \frac{2g_{11}}{g_1^4\sum_{k\neq1}g_{kk}}\left(g_{11}g_{mm}^2 + \sum_{i\neq1}g_{ii}g_{mm}^2 - g_{mm}^3\right) \text{ for } m \neq 1.
\end{aligned} \tag{4.2.27}$$

For a_{ij} in (3.1.2), let us define an operator LX as follows.

$$
\begin{aligned}
LX := {} & X_t - \sum_{i,j=1}^{n} a_{ij} D_{ij} X - \frac{2}{I^{3/2}} g \sum_{i\neq 1} g_{ii1} X_1 - \frac{3}{I^{3/2}} g_1 \sum_{j\neq 1} g_{jj} X_1 \\
& - \frac{1}{2I^{5/2}}\Big(- I g^2 \Big(g g_1 R_{g,2} + 4 g g_1 g_{11} \sum_{j\neq 1} g_{jj}\Big) - 6 g^2 g_1^3 \sum_{j\neq 1} g_{jj} + 6 g^5 g_1^3 g_{11} \sum_{j\neq 1} g_{jj}\Big) X_1 \\
& + \frac{3}{I^{5/2}} (I g \sum_{j\neq 1} g_{jj} + g g_{11} + g_1^2) g^2 g_1 g_{11} X_1 - \frac{3}{I^{5/2}} g^3 g_1 g_{11}^2 X_1 - \frac{3}{I^{5/2}} (g^3 g_1^2 \sum_{j\neq 1} g_{jj} + g_1^2) g^2 g_1 g_{11} X_1 \\
& + \frac{3}{I^{5/2}} (I g \sum_{j\neq 1} g_{jj} + g g_{11} + g_1^2) g g_1^3 X_1 - \frac{1}{I^{3/2}} (2 g^2 g_1^2 \sum_{j\neq 1} g_{jj} + I \sum_{j\neq 1} g_{jj} + g_{11}) g_1 X_1 \\
& - \frac{3}{I^{5/2}} g^2 g_1^3 g_{11} X_1 - \frac{3}{I^{5/2}} (g^3 g_1^2 \sum_{j\neq 1} g_{jj} + g_1^2) g g_1^2 g_1 X_1 + \frac{1}{I^{3/2}} (2 g^2 g_1^2 + I) g_1 g_{11} X_1 \\
& + \frac{3}{I^{3/2}} g^2 g_1^3 \sum_{j\neq 1} g_{jj} X_1 - \frac{3}{I^{3/2}} g^2 g_1^3 g_{11} X_1 + \frac{3}{I^{5/2}} g^3 g_1 g_{11} \sum_{j\neq 1} g_{jj} X_1 + \frac{3}{I^{5/2}} g^2 g_1^3 \sum_{j\neq 1} g_{jj} X_1 \\
& - \Big(2 b g_1^3 \exp{(b g_1^2)} - \frac{2 g_{11}}{g_1}\Big) \frac{1}{I^{3/2}} g \sum_{j\neq 1} g_{jj} X_1 + \frac{2}{I^{3/2}} g g_1^2 \sum_{n\neq 1} X_m^2 .
\end{aligned}
$$

At the interior minimum point $P_0(t)$ of X, we have $X_i = 0$ and $X_{ii} \geq 0$ for all $i = 1, \ldots, n$ so that

$$
\frac{1}{I^{3/2}} g \sum_{i\neq 1} g_{ii} X_{11} + \frac{1}{I^{3/2}} \sum_{m\neq 1} (I g \sum_{j\neq 1, m} g_{jj} + g g_{11} + g_1^2) X_{mm} \geq 0 \tag{4.2.28}
$$

and $X_i = 0$ for all i at the minimum point $P_0 = P_0(t)$ of X. So $X_t \geq LX$ at P_0. Since $g_{11} \leq 0$ and $R_{g,2} \leq 0$, the following group of terms G, which appears in the evolution of X

$$
\begin{aligned}
G := {} & \frac{1}{g_1^2} \frac{1}{2I^{5/2}} (I R_{g,2} - 6 g^2 g_1^2 g_{11} \sum_{j\neq 1} g_{jj} - 6 g g_1^4 \sum_{j\neq 1} g_{jj}) g_{11} \\
& + \frac{1}{g_1^2} \frac{1}{2I^{5/2}} g_{11}^2 \Big(4I \sum_{j\neq 1} g_{jj} - I g^3 R_{g,2} - 4 I g^3 g_{11} \sum_{j\neq 1} g_{jj} - 2 g^4 g_1 (g g_1 R_{g,2} + 4 g g_1 g_{11} \sum_{j\neq 1} g_{jj}) \\
& - 10 g^2 g_1^2 \sum_{j\neq 1} g_{jj} + 18 g^5 g_1^2 \sum_{j\neq 1} g_{jj} g_{11}\Big) \\
& - \frac{5}{2I^{7/2}} \Big(4 I g_1 \sum_{j\neq 1} g_{jj} - I g^2 (g g_1 R_{g,2} + 4 g g_1 g_{11} \sum_{j\neq 1} g_{jj}) - 6 g^2 g_1^3 \sum_{j\neq 1} g_{jj} + 6 g^5 g_1^3 g_{11} \sum_{j\neq 1} g_{jj}\Big) g g_1 g_{11} \\
& + \frac{5}{2I^{7/2}} g_1^2 (- I g R_{g,2} + 6 g^3 g_1^2 g_{11} \sum_{j\neq 1} g_{jj}) + \frac{15}{I^{7/2}} g^2 g_1^6 \sum_{j\neq 1} g_{jj} - \frac{20}{I^{5/2}} g_1^2 \sum_{j\neq 1} g_{jj} g g_{11} + \frac{1}{I^{5/2}} g^2 R_{g,2} g_{11} \\
& + \frac{1}{I^{5/2}} \Big(- 4 g g_1^2 g_{11} \sum_{j\neq 1} g_{jj} + g g_1^2 (\sum_{i\neq 1} g_{ii})^2 - g g_1^2 \sum_{i\neq 1} g_{ii}^2 \Big) + \frac{1}{I^{3/2}} b g_1^2 \Big((\sum_{i\neq 1} g_{ii})^2 - \sum_{i\neq 1} g_{ii}^2 \Big) \exp{(b g_1^2)} \\
& - \frac{5}{2I^{7/2}} \Big(I (\sum_{i\neq 1} g_{ii})^2 - I \sum_{i\neq 1} g_{ii}^2 - 6 g^2 g_1^2 g_{11} \sum_{j\neq 1} g_{jj}\Big) g^2 g_{11}
\end{aligned}
$$

satisfies

$$
\begin{aligned}
G =& \frac{1}{g_1^2}\frac{1}{2I^{5/2}}\Big(IR_{g,2}+O(g)\Big)g_{11} + \frac{1}{g_1^2}\frac{1}{2I^{5/2}}g_{11}^2\Big(4I\sum_{j\neq 1} g_{jj} + O(g)\Big) - \frac{5}{2I^{7/2}}\Big(4Ig_1\sum_{j\neq 1} g_{jj} + O(g)\Big)gg_1g_{11} \\
&+ \frac{5}{2I^{7/2}}g_1^2(-IgR_{g,2}+O(g)) + \frac{1}{I^{5/2}}\Big(-4gg_1^2g_{11}\sum_{j\neq 1} g_{jj} + O(g)\Big) - \frac{5}{2I^{7/2}}\Big(I\sum_{i,j\neq 1\ ,i\neq j} g_{ii}g_{jj} + O(g)\Big)g^2g_{11} \\
&+ \frac{1}{I^{5/2}}g^2R_{g,2}g_{11} - \frac{20}{I^{5/2}}g_1^2\sum_{j\neq 1} g_{jj}gg_{11} + \frac{15}{I^{7/2}}g^2g_1^6\sum_{j\neq 1} g_{jj} + \frac{1}{I^{3/2}}bg_1^2\Big((\sum_{i\neq 1} g_{ii})^2 - \sum_{i\neq 1} g_{ii}^2\Big)\exp(bg_1^2)
\end{aligned}
$$

and is nonnegative when $g<\varepsilon_0'$, for some constants $0<\varepsilon_0'<\varepsilon_0$.

Since we have $X=\frac{g_{11}}{g_1^2}+\exp(bg_1^2)$ and $X^2=\frac{g_{11}^2}{g_1^4}+\frac{2g_{11}}{g_1^2}\exp(bg_1^2)+\exp(2bg_1^2)$ at P_0, we get $g_{11}=g_1^2X-g_1^2\exp(bg_1^2)$ and $g_{11}^2=g_1^4X^2-2g_1^2\exp(bg_1^2)g_1^2X+g_1^4\exp(2bg_1^2)$ so that

$$
\begin{aligned}
LX =& G + \frac{1}{g_1^2\sum_{k\neq 1} g_{kk}}\frac{1}{I^{3/2}}2(gg_{11}+g_1^2)\sum_{i,j\neq 1;i\neq j} g_{ij1}^2 + \frac{1}{g_1^2\sum_{k\neq 1} g_{kk}}\frac{1}{I^{3/2}}\Big((gg_{11}+g_1^2) - Ig\sum_{l\neq 1} g_{ll}\Big)\sum_{i\neq 1} g_{ii1}^2 \\
&+ \frac{1}{g_1^2\sum_{k\neq 1} g_{kk}}\frac{1}{I^{3/2}}Ig\Big(\sum_{i,j\neq 1;i\neq j;}\ \sum_{l\neq 1,i,j} g_{ll}g_{ij1}^2 + 2\sum_{i,j\neq 1;i\neq j} g_{jj}g_{ii1}^2\Big) \\
&+ \frac{1}{4g_1^2}\frac{1}{\sqrt{I}}g(\sum_{i\neq 1} g_{ii1})^2 + \frac{1}{3g_1^2\sum_{k\neq 1} g_{kk}}\frac{1}{I^{3/2}}(gg_{11}+g_1^2)\sum_{i\neq 1} g_{ii1}^2 \\
&+ \frac{g}{4g_1^2\sqrt{I}}\Big(\Big(\sum_{i\neq 1} g_{ii1} - \frac{4g^2g_1g_{11}}{I}\sum_{j\neq 1} g_{jj}\Big)^2 + \Big(\sum_{i\neq 1} g_{ii1} - \frac{12g_1}{I^2}f_{11}^2\Big)^2 + \Big(\sum_{i\neq 1} g_{ii1} - \frac{8bg_1g_{11}}{I}\exp(bg_1^2)\Big)^2\Big) \\
&+ \frac{1}{12g_1^2\sum_{k\neq 1} g_{kk}}\frac{1}{I^{3/2}}f_{11}\sum_{i\neq 1}\Big(\Big(g_{ii1} + \frac{12}{f_{11}}g^3g_1g_{11}\sum_{j\neq 1} g_{jj}g_{ii}\Big)^2 + \Big(g_{ii1} + \frac{24g_{11}}{g_1f_{11}}Igg_{ii}^2\Big)^2 \\
&+ \Big(g_{ii1} - \frac{24}{g_1}g_{11}g_{ii}\Big)^2 + \Big(g_{ii1} - \frac{12}{f_{11}}g_1\sum_{j\neq 1} g_{jj}g_{ii}\Big)^2 + \Big(g_{ii1} - \frac{24g_{11}}{g_1f_{11}}Ig\sum_{j\neq 1} g_{jj}g_{ii}\Big)^2 \\
&+ \Big(g_{ii1} + \frac{12}{f_{11}}g_1(\sum_{j\neq 1} g_{jj})^2\Big)^2 + \Big(g_{ii1} + \frac{36}{f_{11}}g_1g_{11}\sum_{j\neq 1} g_{jj}\Big)^2 + \Big(g_{ii1} + \frac{24}{g_1f_{11}}gg_{11}^2\sum_{j\neq 1} g_{jj}\Big)^2\Big) \\
&- \frac{1}{I^{5/2}}\Big(6bg^3g_1^8\exp(bg_1^2) + gg_1^2\Big((4+10g^2g_1^2)\sum_{j\neq 1} g_{jj} - 6gg_1^4\Big)\Big)X^3 \\
&- \frac{16b^2}{I^{5/2}}gg_1^4\exp(4bg_1^2)X^2 - \frac{1}{I^{5/2}}(16b^2gg_1^4 - 18bg^3g_1^8)\exp(2bg_1^2)X^2 \\
&- \frac{1}{I^{3/2}}gg_1^4\sum_{i\neq 1} g_{ii}(4b^2g_1^2+10b)\exp(bg_1^2)X^2 + \frac{6b}{I^{5/2}}(g\sum_{j\neq 1} g_{jj} + gg_{11} + g_1^2)g^2g_1^6\exp(bg_1^2)X^2 \\
&+ \frac{1}{I^{5/2}}\Big(-b(12g^2g_1^8 - 2g^4g_1^{10}) + (6bg^3g_1^6 + 12Igg_1^2)\sum_{j\neq 1} g_{jj} + 18(-g^2g_1^6 + g^3g_1^4\sum_{j\neq 1} g_{jj})\Big)\exp(bg_1^2)X^2 \\
&- \frac{1}{I^{3/2}}\frac{48(n-1)}{(gg_{11}+g_1^2)}g^2g_{11}^2\sum_{j\neq 1} g_{jj}X^2 - \frac{1}{\sum_{k\neq 1} g_{kk}}\frac{48}{I^{3/2}}g_1(gg_{11}+g_1^2)\sum_{i\neq 1} g_{ii}^2X^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{I^{3/2}}\frac{108(n-1)}{(gg_{11}+g_1^2)}g_1^4\sum_{j\neq 1}g_{jj}X^2+\frac{1}{I^{3/2}}\Big(6(gg_{11}+g_1^2)+2Ig_1^2\Big)\sum_{j\neq 1}g_{jj}X^2\\
&+\frac{1}{I^{3/2}}\sum_{i\neq 1}(Ig\sum_{j\neq 1,i}g_{jj}+gg_{11}+g_1^2)\frac{2}{\sum_{k\neq 1}g_{kk}}g_{ii}^2X^2\\
&-\frac{1}{I^{5/2}}g_1\Big(\Big(4Ig_1-6g^2g_1(g_1^2+gg_{11})+2Ig^3g_1g_{11}\Big)\sum_{j\neq 1}g_{jj}-Ig^3g_1R_{g,2}\Big)X^2\\
&+\frac{1}{I^{5/2}}\Big(-6g^2g_1^4g_{11}-2Ig^2g_1^4\sum_{j\neq 1}g_{jj}-23g^2g_1^4\sum_{j\neq 1}g_{jj}-6g^2g_1^2(gg_{11}+g_1^2)\sum_{j\neq 1}g_{jj}\Big)X^2\\
&-\frac{48}{I^{3/2}}\frac{1}{(gg_{11}+g_1^2)}I^2g^2\frac{1}{\sum_{k\neq 1}g_{kk}}\Big(\sum_{i\neq 1}g_{ii}^4+(\sum_{j\neq 1}g_{jj})^2\sum_{i\neq 1}g_{ii}^2\Big)X^2\\
&+\frac{1}{I^{5/2}}g_1\Big(-Ig^2\Big(gg_1R_{g,2}+4gg_1g_{11}\sum_{j\neq 1}g_{jj}\Big)+6g^5g_1^3g_{11}\sum_{j\neq 1}g_{jj}\Big)X^2\\
&-\frac{5}{2I^{7/2}}\Big(4I\sum_{j\neq 1}g_{jj}-Ig^2(gR_{g,2}+4gg_{11}\sum_{j\neq 1}g_{jj})-6g^2g_1^2\sum_{j\neq 1}g_{jj}+6g^5g_1^2g_{11}\sum_{j\neq 1}g_{jj}\Big)g^2g_1^4X^2\\
&-\frac{1}{I^{5/2}}\Big(4g^5g_1^4(\sum_{j\neq 1}g_{jj})^2+\frac{1}{(gg_{11}+g_1^2)}12g^6g_1^4\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2\Big)X^2\\
&+\frac{1}{I^{5/2}}(32b^2gg_1^4-18bg^3g_1^8)\exp(3bg_1^2)X\\
&+\frac{1}{I^{5/2}}\Big(Igg_1^4(8b^2g_1^2+20b)\sum_{j\neq 1}g_{jj}+12bg^2g_1^6(g_1^2-gg_{11}-2g\sum_{j\neq 1}g_{jj})\Big)\exp(2bg_1^2)X\\
&-\frac{1}{I^{5/2}}\Big((12g_1^2+30g^2g_1^4)\sum_{j\neq 1}g_{jj}-18gg_1^6\Big)g\exp(2bg_1^2)X\\
&+\frac{b}{I^{5/2}}\Big((-2gg_1^2-4g^2g_{11}+2g^4g_1^2g_{11})\sum_{j\neq 1}g_{jj}-Ig^2R_{g,2}\Big)gg_1^4\exp(bg_1^2)X\\
&+\frac{b}{I^{5/2}}gg_1^6\Big(6gg_{11}+(2g-12g^3g_1^2)\sum_{j\neq 1}g_{jj}\Big)\exp(bg_1^2)X\\
&-\frac{b}{I^{5/2}}\Big(-I\Big(gg_1R_{g,2}+4gg_1g_{11}\sum_{j\neq 1}g_{jj}\Big)+6g^3g_1^3g_{11}\sum_{j\neq 1}g_{jj}\Big)g^2g_1^3\exp(bg_1^2)X\\
&+\frac{2}{I^{5/2}}g_1\Big(\Big(4Ig_1-6g^2g_1(g_1^2+gg_{11})+2Ig^3g_1g_{11}\Big)\sum_{j\neq 1}g_{jj}-Ig^3g_1R_{g,2}\Big)\exp(bg_1^2)X\\
&+\frac{1}{I^{3/2}}\frac{n-1}{(gg_{11}+g_1^2)}(96g^2g_{11}^2+216g_1^4)\sum_{j\neq 1}g_{jj}\exp(bg_1^2)X\\
&+\frac{1}{I^{3/2}}(gg_{11}+g_1^2)\Big(\frac{96g_1}{\sum_{k\neq 1}g_{kk}}\sum_{i\neq 1}g_{ii}^2-12\sum_{j\neq 1}g_{jj}\Big)\exp(bg_1^2)X\\
&+\frac{1}{I^{3/2}}2g_1^2\sum_{i\neq 1}g_{ii}\exp(bg_1^2)X
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{I^{3/2}}\sum_{i\neq 1}(Ig\sum_{j\neq 1,i}g_{jj}+gg_{11}+g_1^2)\frac{4}{\sum_{k\neq 1}g_{kk}}g_{ii}^2\exp(bg_1^2)X\\
&-\frac{4}{I^{3/2}}(3g^2g_1^2+1)g_1^2\sum_{j\neq 1}g_{jj}\exp(bg_1^2)X+\frac{12}{I^{5/2}}\Big(g(gg_{11}+g_1^2)\sum_{j\neq 1}g_{jj}+gg_1^2g_{11}\Big)gg_1^2\exp(bg_1^2)X\\
&+\frac{96}{I^{3/2}}\frac{1}{(gg_{11}+g_1^2)}I^2g^2\Big(\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2+\frac{1}{\sum_{k\neq 1}g_{kk}}\sum_{i\neq 1}g_{ii}^4\Big)\exp(bg_1^2)X\\
&-\frac{2}{I^{5/2}}g_1\Big(-Ig^2\Big(gg_1R_{g,2}+4gg_1g_{11}\sum_{j\neq 1}g_{jj}\Big)+6g^5g_1^3g_{11}\sum_{j\neq 1}g_{jj}\Big)\exp(bg_1^2)X\\
&+\frac{5}{I^{7/2}}\Big(4I\sum_{j\neq 1}g_{jj}-Ig^2(gR_{g,2}+4gg_{11}\sum_{j\neq 1}g_{jj})-6g^2g_1^2\sum_{j\neq 1}g_{jj}+6g^5g_1^2g_{11}\sum_{j\neq 1}g_{jj}\Big)g^2g_1^4\exp(bg_1^2)X\\
&+\frac{1}{I^{5/2}}\Big(g^2g_1^4(46+12g^2g_1^2)\sum_{j\neq 1}g_{jj}+8g^5g_1^4(\sum_{j\neq 1}g_{jj})^2+\frac{1}{(gg_{11}+g_1^2)}24Ig^6g_1^4\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2\Big)\exp(bg_1^2)X\\
&+\frac{2}{I^{3/2}}\sum_{i\neq 1}(Ig\sum_{j\neq 1,i}g_{jj}+gg_{11}+g_1^2)\frac{1}{g_1^2\sum_{k\neq 1}g_{kk}}\sum_{l\neq 1,i}g_{ll}g_{ii}^2X\\
&+\frac{1}{I^{7/2}}\Big(-I^2R_{g,2}+6Igg_1^2(g_1^2+gg_{11})\sum_{j\neq 1}g_{jj}+15g^3g_1^6\sum_{j\neq 1}g_{jj}\Big)X\\
&+\frac{1}{2I^{5/2}}\Big(-2(1+2g^2g_1^2)(gR_{g,2}+4gg_{11}\sum_{j\neq 1}g_{jj})-IgR_{g,2}-4Igg_{11}\sum_{j\neq 1}g_{jj}+30g^3g_1^2g_{11}\sum_{j\neq 1}g_{jj}\Big)gg_1^2X\\
&+\frac{b}{I^{5/2}}(2-10g^2g_1^2)g_1^4\sum_{j\neq 1}g_{jj}\exp(2bg_1^2)\\
&-\frac{36}{I^{9/2}}g(gg_{11}+g_1^2)^4+\frac{6b}{I^{5/2}}g^3g_1^8\exp(4bg_1^2)\\
&+\Big(-\frac{1}{I^{3/2}}gg_1^4(4b^2g_1^2+10b)\sum_{j\neq 1}g_{jj}+\frac{6b}{I^{5/2}}(2g\sum_{j\neq 1}g_{jj}+gg_{11})g^2g_1^6\Big)\exp(3bg_1^2)\\
&+\Big(-\frac{2b}{I^{5/2}}g^2g_1^8+\frac{1}{I^{5/2}}\Big((4gg_1^2+10g^3g_1^4)\sum_{j\neq 1}g_{jj}-6g^2g_1^6\Big)\Big)\exp(3bg_1^2)\\
&-\frac{6b}{I^{5/2}}(g\sum_{j\neq 1}g_{jj}+gg_{11})gg_1^6\exp(2bg_1^2)\\
&+\frac{b}{I^{5/2}}\Big(-I\Big(gg_1R_{g,2}+4gg_1g_{11}\sum_{j\neq 1}g_{jj}\Big)+6g^3g_1^3g_{11}\sum_{j\neq 1}g_{jj}\Big)g^2g_1^3\exp(2bg_1^2)\\
&-\frac{b}{I^{5/2}}\Big(\Big(-6g_1(g_1^2+gg_{11})+2Igg_1g_{11}\Big)\sum_{j\neq 1}g_{jj}-Igg_1R_{g,2}\Big)g^2g_1^3\exp(2bg_1^2)\\
&+\frac{1}{I^{3/2}}\Big(-\frac{48}{\sum_{k\neq 1}g_{kk}}g_1f_{11}\sum_{i\neq 1}g_{ii}^2-\frac{108(n-1)}{f_{11}}g_1^4\sum_{j\neq 1}g_{jj}+6gg_{11}\sum_{i\neq 1}g_{ii}\Big)\exp(2bg_1^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{I^{3/2}}\sum_{i\neq 1}(Ig\sum_{j\neq 1,i}g_{jj}+f_{11})\frac{2}{\sum_{k\neq 1}g_{kk}}g_{ii}^2\exp(2bg_1^2)\\
&+\frac{6}{I^{5/2}}(1+Ig^2g_1^2)g_1^2\sum_{j\neq 1}g_{jj}\exp(2bg_1^2)\\
&-\frac{1}{I^{5/2}}g_1\Big(\big(4Ig_1-6g^2g_1(g_1^2+gg_{11})+2Ig^3g_1g_{11}\big)\sum_{j\neq 1}g_{jj}-Ig^3g_1R_{g,2}\Big)\exp(2bg_1^2)\\
&+\Big(-\frac{1}{I^{3/2}}\frac{48(n-1)}{(gg_{11}+g_1^2)}g^2g_{11}^2\sum_{j\neq 1}g_{jj}+\frac{6}{I^{5/2}}(g^2g_1^4g_{11}+g^2g_1^2f_{11}\sum_{j\neq 1}g_{jj})\Big)\exp(2bg_1^2)\\
&+\frac{1}{I^{5/2}}g_1\Big(-Ig^2\big(gg_1R_{g,2}+4gg_1g_{11}\sum_{j\neq 1}g_{jj}\big)+6g^5g_1^3g_{11}\sum_{j\neq 1}g_{jj}\Big)\exp(2bg_1^2)\\
&+\Big(-\frac{1}{I^{5/2}}(23+6g^2g_1^2)g_1^4-\frac{48}{I^{3/2}}\frac{1}{f_{11}}I^2\big(\frac{1}{(\sum_{k\neq 1}g_{kk})^2}\sum_{i\neq 1}g_{ii}^4+\sum_{i\neq 1}g_{ii}^2\big)\Big)g^2\sum_{j\neq 1}g_{jj}\exp(2bg_1^2)\\
&-\frac{5}{2I^{7/2}}\Big((4-2g^2g_1^2+6g^5g_1^2g_{11})\sum_{j\neq 1}g_{jj}-Ig^2(gR_{g,2}+4gg_{11}\sum_{j\neq 1}g_{jj})\Big)g^2g_1^4\exp(2bg_1^2)\\
&-\frac{1}{I^{5/2}}\Big(4\sum_{i\neq 1}g_{ii}+\frac{12}{f_{11}}Ig\sum_{i\neq 1}g_{ii}^2\Big)g^5g_1^4\sum_{j\neq 1}g_{jj}\exp(2bg_1^2)\\
&-\frac{b}{I^{5/2}}\Big(2I\sum_{i\neq 1}(Ig\sum_{j\neq 1,i}g_{jj}+f_{11})g_{ii}^2+6gg_1^4f_{11}\sum_{j\neq 1}g_{jj}\Big)\exp(bg_1^2)\\
&+\frac{2}{I^{3/2}}\frac{1}{g_1^2}\sum_{i\neq 1}(Ig\sum_{j\neq 1,i}g_{jj}+f_{11})\Big(\frac{1}{\sum_{k\neq 1}g_{kk}}g_{ii}^3-g_{ii}^2\Big)\exp(bg_1^2)\\
&+\frac{1}{I^{3/2}}\sum_{i,j\neq 1,\,i\neq j}g_{ii}g_{jj}\exp(bg_1^2)-\frac{1}{I^{5/2}}6gg_1^2(g_1^2+gg_{11})\sum_{j\neq 1}g_{jj}\exp(bg_1^2)\\
&-\frac{1}{2I^{5/2}}\Big(-2(1+2g^2g_1^2)(gR_{g,2}+4gg_{11}\sum_{j\neq 1}g_{jj})-IgR_{g,2}-4Igg_{11}\sum_{j\neq 1}g_{jj}\Big)gg_1^2\exp(bg_1^2)\\
&-\frac{15}{I^{5/2}}g^4g_1^4g_{11}\sum_{j\neq 1}g_{jj}\exp(bg_1^2)-\frac{15}{I^{7/2}}g^3g_1^6\sum_{j\neq 1}g_{jj}\exp(bg_1^2)\\
&-\frac{12}{I^{3/2}}\frac{1}{f_{11}}\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2-\frac{1}{I^{3/2}}\frac{12(n-1)}{f_{11}}(\sum_{j\neq 1}g_{jj})^3-\frac{3}{I^{5/2}}g_1^4\sum_{j\neq 1}g_{jj}.
\end{aligned}
$$

and it satisfies

$$
\begin{aligned}
LX=G&+\frac{1}{g_1^2\sum_{k\neq 1}g_{kk}}\frac{1}{I^{3/2}}2(gg_{11}+g_1^2)\sum_{i,j\neq 1;i\neq j}g_{ij1}^2+\frac{1}{g_1^2\sum_{k\neq 1}g_{kk}}\frac{1}{I^{3/2}}\Big((gg_{11}+g_1^2)-Ig\sum_{l\neq 1}g_{ll}\Big)\sum_{i\neq 1}g_{ii1}^2\\
&+\frac{1}{g_1^2\sum_{k\neq 1}g_{kk}}\frac{1}{I^{3/2}}Ig\Big(\sum_{i,j\neq 1;i\neq j;}\sum_{l\neq 1,i,j}g_{ll}g_{ij1}^2+2\sum_{i,j\neq 1;i\neq j}g_{jj}g_{ii1}^2\Big)\\
&+\frac{1}{4g_1^2}\frac{1}{\sqrt{I}}g(\sum_{i\neq 1}g_{ii1})^2+\frac{1}{3g_1^2\sum_{k\neq 1}g_{kk}}\frac{1}{I^{3/2}}(gg_{11}+g_1^2)\sum_{i\neq 1}g_{ii1}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{g}{4g_1^2\sqrt{I}}\Big(\big(\sum_{i\neq 1} g_{ii1}-\frac{4g^2g_1g_{11}}{I}\sum_{j\neq 1} g_{jj}\big)^2+\big(\sum_{i\neq 1} g_{ii1}-\frac{12g_1}{I^2}f_{11}^2\big)^2+\big(\sum_{i\neq 1} g_{ii1}-\frac{8bg_1g_{11}}{I}\exp{(bg_1^2)}\big)^2\Big)\\
&+\frac{1}{12g_1^2\sum_{k\neq 1}g_{kk}}\frac{1}{I^{3/2}}f_{11}\sum_{i\neq 1}\Big(\Big(g_{ii1}+\frac{12}{f_{11}}g^3g_1g_{11}\sum_{j\neq 1}g_{jj}g_{ii}\Big)^2+\Big(g_{ii1}+\frac{24g_{11}}{g_1f_{11}}Igg_{ii}^2\Big)^2\\
&+\Big(g_{ii1}-\frac{24}{g_1}g_{11}g_{ii}\Big)^2+\Big(g_{ii1}-\frac{12}{f_{11}}g_1\sum_{j\neq 1}g_{jj}g_{ii}\Big)^2+\Big(g_{ii1}-\frac{24g_{11}}{g_1f_{11}}Ig\sum_{j\neq 1}g_{jj}g_{ii}\Big)^2\\
&+\Big(g_{ii1}+\frac{12}{f_{11}}g_1(\sum_{j\neq 1}g_{jj})^2\Big)^2+\Big(g_{ii1}+\frac{36}{f_{11}}g_1g_{11}\sum_{j\neq 1}g_{jj}\Big)^2+\Big(g_{ii1}+\frac{24}{g_1f_{11}}gg_{11}^2\sum_{j\neq 1}g_{jj}\Big)^2\Big)\\
&+\Big(\frac{b}{I^{5/2}}(2-10g^2g_1^2)g_1^4\sum_{j\neq 1}g_{jj}+O(1)\Big)\exp{(2bg_1^2)}-\frac{1}{I^{5/2}}\Big(6bg^3g_1^8\exp{(bg_1^2)}+O(g)\Big)X^3\\
&+\frac{1}{I^{5/2}}\Big(b^2O(g)\exp{(4bg_1^2)}+(b^2O(g)+bO(g^3))\exp{(2bg_1^2)}+\Big(O(g)+b^2O(g^2)+bO(g^2)\Big)\exp{(bg_1^2)}\Big)X^2\\
&+\frac{1}{I^{5/2}}\Big(O(1)+O(g)\Big)X^2+\frac{1}{I^{5/2}}\Big(O(1)+O(g)+bO(g)+b^2O(g)\Big)\exp{(bg_1^2)}X\\
&+\frac{2}{I^{3/2}}\sum_{i\neq 1}(Ig\sum_{j\neq 1,i}g_{jj}+f_{11})\frac{1}{g_1^2\sum_{k\neq 1}g_{kk}}\sum_{l\neq 1,i}g_{ll}g_{ii}^2X+\frac{1}{I^{5/2}}\Big(-I^2R_{g,2}+O(g)\Big)X\\
&+\frac{6b}{I^{5/2}}g^3g_1^8\exp{(4bg_1^2)}+\frac{1}{I^{5/2}}\Big(O(g)+b^2O(g)+bO(g)\Big)\exp{(3bg_1^2)}\\
&+\frac{1}{I^{5/2}}\Big(bO(g)+bO(g^2)+6g^2g_1^4g_{11}+O(g^2)\Big)\exp{(2bg_1^2)}\\
&+\frac{1}{I^{5/2}}\Big(-b\Big(2I\sum_{i\neq 1}(Ig\sum_{j\neq 1,i}g_{jj}+f_{11})g_{ii}^2+6gg_1^4f_{11}\sum_{j\neq 1}g_{jj}\Big)+O(1)+O(g)\Big)\exp{(bg_1^2)}\\
&-\frac{12}{I^{3/2}}\frac{1}{f_{11}}\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2-\frac{1}{I^{3/2}}\frac{12(n-1)}{f_{11}}(\sum_{j\neq 1}g_{jj})^3-\frac{3}{I^{5/2}}g_1^4\sum_{j\neq 1}g_{jj}-\frac{36}{I^{9/2}}g(gg_{11}+g_1^2)^4.
\end{aligned}
$$

Suppose that there exists some time $t_0>0$ such that $X(P_0(t_0),t_0)=0$ for the first time. Then

$$
\begin{aligned}
0\geq X_t\geq&\Big(\frac{2b}{I^{5/2}}g_1^4\sum_{j\neq 1}g_{jj}+O(1)\Big)\exp{(2bg_1^2)}\\
&+\frac{1}{I^{5/2}}\Big(O(g)+bO(g^2)+bO(g^3)\exp{(bg_1^2)}\Big)X^3+\frac{1}{I^{5/2}}\Big(O(1)+O(g)\Big)X^2\\
&+\frac{1}{I^{5/2}}\Big(b^2O(g)\exp{(4bg_1^2)}+(b^2O(g)+bO(g^3))\exp{(2bg_1^2)}+\Big(O(g)+b^2O(g^2)+bO(g^2)\Big)\exp{(bg_1^2)}\Big)X^2\\
&+\frac{1}{I^{5/2}}\Big(O(1)+O(g)+bO(g)+b^2O(g)\Big)\exp{(bg_1^2)}X+\frac{1}{I^{5/2}}O(g)X\\
&+\frac{1}{I^{5/2}}\Big(O(g)+b^2O(g)+bO(g)\Big)\exp{(3bg_1^2)}+\frac{1}{I^{5/2}}\Big(bO(g)+bO(g^2)+O(g)\Big)\exp{(2bg_1^2)}\\
&+\frac{1}{I^{5/2}}\Big(bO(1)+O(1)+O(g)\Big)\exp{(bg_1^2)}+O(1)
\end{aligned}
$$

$$\geq \frac{b}{I^{5/2}} g_1^4 \sum_{j\neq 1} g_{jj} \exp(2bg_1^2) > 0$$

for all $g < \varepsilon_0'$, for some constants $C > 0$, $0 < \varepsilon_0' < \varepsilon_0$ and for sufficiently large $b > 0$ at the time t_0, which is a contradiction. Hence, X cannot become zero in the interior.

Now, it is sufficient to check X on the interface $\Gamma(t)$ where $g = 0$. Suppose that X gets a minimum at a point $P_0 = P_0(t) \in \Gamma(t)$. At P_0, we choose a coordinate system where $e_1 = \nu$ and $g_{ijl} = 0$ for $i \neq j$. Then

$$X = \frac{g_{\nu\nu}}{|\nabla g|^2} + \exp(b|\nabla g|^2) = \frac{g_{11}}{g_1^2} + \exp(bg_1^2), \tag{4.2.29}$$

and we take $b > 0$ sufficiently large such that $X > 0$ at time $t = 0$. We will determine the value of the constant b later. At $P_0 \in \Gamma(t)$, the first order derivatives of X satisfy

$$0 \leq X_1 = \frac{1}{g_1^2} g_{111} + \left(2bg_1 \exp(bg_1^2) - \frac{2g_{11}}{g_1^3}\right) g_{11}, \ 0 = X_m = \frac{1}{g_1^2} g_{11m} \text{ for } m \neq 1 \tag{4.2.30}$$

so that $g_{111} \geq \left(-2bg_1^3 \exp(bg_1^2) + \frac{2g_{11}}{g_1}\right) g_{11}$ and $g_{11m} = 0$ for $m \neq 1$. Also, $X_{mm} \geq 0$ for $m \neq 1$ at $P_0(t)$. However, the evolution of $X(P_0(t), t)$ on the free boundary is affected by the motion of the free boundary too. For any point $\gamma(t)$ on the $(n-1)$-dimensional free-boundary hypersurface $\Gamma(t)$ of the flow, we have

$$\frac{d}{dt} X(\gamma(t), t) = X_t + \nabla_x X \cdot \dot{\gamma}(t). \tag{4.2.31}$$

Since $g(\gamma(t), t) = 0$ so that $g_t = -\nabla_x g \cdot \dot{\gamma}(t) = -g_1 \dot{\gamma}_1(t)$ at $P_0(t) = \gamma(t)$, the term $\nabla_x X \cdot \dot{\gamma}(t)$ is given by

$$\nabla_x X \cdot \dot{\gamma}(t) = X_1 \dot{\gamma}_1(t) = -\frac{g_t}{g_1} X_1 = -\frac{1}{I^{3/2}} g_1 \sum_{j\neq 1} g_{jj} X_1, \tag{4.2.32}$$

at $P_0(t) = \gamma(t) \in \Gamma(t)$. Then the operator LX satisfies at $P_0(t) \in \Gamma(t)$

$$\begin{aligned}
\frac{d}{dt} X(P_0(t), t) \geq LX &= X_t - \frac{1}{I^{3/2}} \sum_{m\neq 1} g_1^2 X_{mm} - \frac{1}{I^{3/2}} g_1 \sum_{j\neq 1} g_{jj} X_1 = \frac{d}{dt} X(\gamma(t), t) - \frac{1}{I^{3/2}} \sum_{m\neq 1} g_1^2 X_{mm} \\
= &\frac{3}{I^{3/2}} g_1 \sum_{j\neq 1} g_{jj} X_1 + \frac{2}{\sum_{k\neq 1} g_{kk}} \frac{1}{I^{3/2}} \sum_{\substack{i,j\neq 1 \\ i\neq j}} g_{ij1}^2 + \frac{5}{3\sum_{k\neq 1} g_{kk}} \frac{1}{I^{3/2}} \sum_{i\neq 1} g_{ii1}^2 + \frac{1}{12\sum_{k\neq 1} g_{kk}} \frac{1}{I^{3/2}} \sum_{i\neq 1} \left(g_{ii1} - \frac{24}{g_1} g_{11} g_{ii}\right)^2 \\
&+ \frac{1}{12\sum_{k\neq 1} g_{kk}} \frac{1}{I^{3/2}} \sum_{i\neq 1} \left(\left(g_{ii1} - \frac{12g_1}{f_{11}} \sum_{j\neq 1} g_{jj} g_{ii}\right)^2 + \left(g_{ii1} + \frac{12g_1}{f_{11}} (\sum_{j\neq 1} g_{jj})^2\right)^2 + \left(g_{ii1} + \frac{36g_1}{f_{11}} g_{11} \sum_{j\neq 1} g_{jj}\right)^2 \right) \\
&+ \frac{1}{g_1^2} \frac{1}{2I^{3/2}} R_{g,2} g_{11} + \frac{1}{g_1^2} \frac{2}{I^{3/2}} g_{11}^2 \sum_{j\neq 1} g_{jj} + \frac{1}{I^{3/2}} b g_1^2 \Big((\sum_{i\neq 1} g_{ii})^2 - \sum_{i\neq 1} g_{ii}^2 \Big) \exp(bg_1^2) \\
&+ \Big(\frac{2b}{I^{5/2}} g_1^4 \sum_{j\neq 1} g_{jj} + O(1) \Big) \exp(2bg_1^2) + \frac{1}{I^{5/2}} \Big(-2bIg_1^2 \sum_{i\neq 1} g_{ii}^2 + O(1) \Big) \exp(bg_1^2) - \frac{12}{g_1^2 I^{3/2}} \sum_{j\neq 1} g_{jj} \sum_{i\neq 1} g_{ii}^2
\end{aligned}$$

$$
\begin{aligned}
&-\frac{12(n-1)}{g_1^2 I^{3/2}}\Big(\sum_{j\neq 1} g_{jj}\Big)^3-\frac{3}{I^{5/2}}g_1^4\sum_{j\neq 1}g_{jj}+\frac{1}{I^{5/2}}O(1)X^2+\frac{1}{I^{5/2}}O(1)\exp\big(bg_1^2\big)X+\frac{2}{I^{3/2}}\sum_{i\neq 1}g_{ii}X-\frac{1}{\sqrt{I}}R_{g,2}X\\
\geq&\Big(\frac{2b}{I^{5/2}}g_1^4\sum_{j\neq 1}g_{jj}+O(1)\Big)\exp\big(2bg_1^2\big)+\frac{1}{I^{5/2}}\Big(-2bIg_1^2\sum_{i\neq 1}g_{ii}^2+O(1)\Big)\exp\big(bg_1^2\big)\\
&+\frac{1}{I^{5/2}}O(1)X^2+\frac{1}{I^{5/2}}O(1)\exp\big(bg_1^2\big)X+\frac{2}{I^{3/2}}\sum_{i\neq 1}g_{ii}X-\frac{1}{\sqrt{I}}R_{g,2}X+O(1)\\
&-\frac{12}{g_1^2 I^{3/2}}\sum_{j\neq 1}g_{jj}\sum_{i\neq 1}g_{ii}^2-\frac{12(n-1)}{g_1^2 I^{3/2}}\Big(\sum_{j\neq 1}g_{jj}\Big)^3-\frac{3}{I^{5/2}}g_1^4\sum_{j\neq 1}g_{jj},
\end{aligned}
$$

the last line of which is uniformly bounded by Lemma 3.2.3 and Corollary 4.2.2. Let us take $b > 0$ sufficiently large, so that $X_0 = X(P_0(0), 0)$ is positive. If $X > 0$ on $\Gamma(t)$ for all time, then $X > 0$ on $\Omega(t)$ because X cannot be zero on the interior points of $\Omega(t)$. Otherwise, there exists some time $t_0 > 0$ such that $X(P_0(t_0), t_0) = 0$ for the first time. Then at $P_0(t_0)$

$$
0 \geq \frac{d}{dt}X(\gamma(t), t) \geq \frac{b}{I^{5/2}}g_1^4\sum_{j\neq 1}g_{jj}\exp\big(2bg_1^2\big) > 0 \tag{4.2.33}
$$

for a sufficiently large $b > 0$, which is a contradiction. Hence X cannot become zero on the boundary as well. This concludes the proof. □

Corollary 4.2.6. *Let us assume the conditions in the subsection 1.4. Then there exists a constant $C > 0$ satisfying*

$$
g_{\nu\nu} \geq -C \tag{4.2.34}
$$

for $0 \leq g \leq 1$ and $0 \leq t \leq T$, where ν is the unit normal to the level set $\Gamma_P(t)$ at the point $P \in \Omega(t)$.

Proof.

$$
2g_{\nu\nu}\sum_{i\neq\nu}g_{ii} = R_{g,2}-\Big(\sum_{i\neq\nu}g_{ii}\Big)^2+\sum_{i,j\neq\nu}g_{ij}^2+2\sum_{i\neq\nu}g_{\nu i}^2 \tag{4.2.35}
$$

and $\sum_{i\neq\nu} g_{ii}$ is uniformly bounded above and below by the inequality (4.2.18). □

We note that a simpler version of the inequality,(4.2.23), has been obtained by Kim, Lee and Rhee (see Lemma 4.4 in [12]) for the flow with a flat side which evolves by the α-th power of Gauss curvature and this is equivalent to our Lemma 4.2.5 in the special case $\alpha = 1$ and $n = 2$.

Chapter 5

Hölder estimates

Throughout the section, we assume that g is a solution of (1.3.8) and it is smooth up to the interface in its support for time $0 \leq t \leq T < T_c$. Around $P = (0, y_0, t) \in \Gamma(t)$, the parabolic box $\mathcal{B}_\eta(P)$ is defined to be $\{(x_{n+1}, (x_2, \ldots, x_n), t) = (z, y, t) | 0 \leq z \leq \eta^2, |y - y_0| \leq \eta, t_0 - \eta^2 \leq t \leq t_0\}$ for $\eta > 0$.

5.1 $C_s^{1,\alpha}$ estimates

Let us consider an arbitrary point $P = (0, y_0, t_0) \in \Gamma(t_0),\ 0 \leq t_0 \leq T$. We will show that the transformed function (1.6.1) $h \in C_s^{1,\alpha}$ near P. As in Daskalopoulos and Hamilton [4], the first-order and second-order derivatives of g are given by

$$\begin{aligned}
&g_t = -\frac{h_t}{h_{n+1}},\ g_1 = \frac{1}{h_{n+1}},\ g_i = -\frac{h_i}{h_{n+1}} \text{ for } i = 2, \ldots, n,\\
&g_{11} = -\frac{1}{h_{n+1}^3} h_{n+1,n+1},\ g_{1i} = -\frac{1}{h_{n+1}}\Big(-\frac{h_i}{h_{n+1}^2} h_{n+1,n+1} + \frac{1}{h_{n+1}} h_{n+1,i}\Big) \text{ for } i = 2, \ldots, n,\\
&g_{ij} = -\frac{1}{h_{n+1}}\Big(\frac{h_i h_j}{h_{n+1}^2} h_{n+1,n+1} - \frac{h_i}{h_{n+1}} h_{n+1,j} - \frac{h_j}{h_{n+1}} h_{n+1,i} + h_{ij}\Big) \text{ for } i, j = 2, \ldots, n.
\end{aligned} \tag{5.1.1}$$

These relations yield that

$$\begin{aligned}
R_{g,2} = \sum_{i,j=1}^{n} (g_{ii} g_{jj} - g_{ij}^2) = &\frac{1}{h_{n+1}^2} \sum_{i,j=2}^{n} \Bigg((h_{ii} h_{jj} - h_{ij}^2) - \Big(\frac{h_i^2}{h_{n+1}^2} h_{n+1,j}^2 + \frac{h_j^2}{h_{n+1}^2} h_{n+1,i}^2 - \frac{2 h_i h_j}{h_{n+1}^2} h_{n+1,i} h_{n+1,j} \Big)\\
&+ \Big(\frac{h_i^2}{h_{n+1}^2} h_{n+1,n+1} h_{jj} + \frac{h_j^2}{h_{n+1}^2} h_{n+1,n+1} h_{ii} - \frac{2 h_i h_j}{h_{n+1}^2} h_{n+1,n+1} h_{ij} \Big)\\
&- 2\Big(\frac{h_i}{h_{n+1}} h_{n+1,i} h_{jj} + \frac{h_j}{h_{n+1}} h_{n+1,j} h_{ii} - \frac{h_i}{h_{n+1}} h_{n+1,j} h_{ij} - \frac{h_j}{h_{n+1}} h_{n+1,i} h_{ij} \Big)\Bigg)\\
&+ \frac{2}{h_{n+1}^4} \sum_{i=2}^{n} (h_{n+1,n+1} h_{ii} - h_{n+1,i}^2),
\end{aligned}$$

$$\sum_{i,j=1}^{n}(g_j^2 g_{ii} + g_i^2 g_{jj} - 2g_i g_j g_{ij}) = -2\sum_{i,j=2}^{n}\frac{1}{h_{n+1}^3}(h_j^2 h_{ii} - h_i h_j h_{ij}) - 2\sum_{i=2}^{n}\frac{1}{h_{n+1}^3}h_{ii},$$

$$\sum_{j\neq i}^{n}(g_i g_{jj} - g_j g_{ij}) = \frac{1}{h_{n+1}^3}h_{n+1,i} + \sum_{j=2}^{n}\Big(\frac{h_j^2}{h_{n+1}^3}h_{n+1,i} - \frac{h_i h_j}{h_{n+1}^3}h_{n+1,j}\Big) + \sum_{j=2}^{n}\Big(\frac{h_i}{h_{n+1}^2}h_{jj} - \frac{h_j}{h_{n+1}^2}h_{ij}\Big) \text{ for } i = 2, \ldots, n,$$

$$\sum_{j\neq i}^{n}(g_j^2 + g g_{jj}) = \frac{h_{n+1}}{h_{n+1}^3} - \frac{x_{n+1}}{h_{n+1}^3}h_{n+1,n+1} + \sum_{j\neq 1,i}^{n} h_j^2\Big(\frac{h_{n+1}}{h_{n+1}^3} - \frac{x_{n+1}}{h_{n+1}^3}h_{n+1,n+1}\Big) - \frac{x_{n+1}}{h_{n+1}}\sum_{j\neq 1,i}^{n}\Big(-2\frac{h_j}{h_{n+1}}h_{n+1,j} + h_{jj}\Big)$$
$$\text{for } i = 2, \ldots, n,$$

$$\begin{aligned}
&\sum_{i,j,k=1}^{n} g_i g_k\Big((g_{ik}g_{jj} - g_{jk}g_{ij})g + g_j^2 g_{ik} + g_i g_k g_{jj} - g_i g_j g_{jk} - g_j g_k g_{ij}\Big)\\
&= -\frac{1}{h_{n+1}^5}\sum_{i=2}^{n} h_{ii} - \frac{1}{h_{n+1}^5}\sum_{i,j=2}^{n} h_i^2 h_{jj} + \frac{x_{n+1}}{h_{n+1}^6}\sum_{i=2}^{n}(h_{ii}h_{n+1,n+1} - h_{n+1,i}^2)\\
&\quad + \frac{x_{n+1}}{h_{n+1}^6}\sum_{i,j=2}^{n} h_i h_j (h_{ij}h_{n+1,n+1} - h_{n+1,i}h_{n+1,j})\\
&\quad - \frac{x_{n+1}}{h_{n+1}^5}\sum_{i,j,k=2}^{n} h_i h_k\Big(h_{jj}(h_i h_{n+1,k} + h_k h_{n+1,i}) - h_{jk}(h_i h_{n+1,j} + h_j h_{n+1,i})\Big)\\
&\quad - \frac{1}{h_{n+1}^5}\sum_{i,j=2}^{n}(h_i^2 h_{jj} - h_i h_j h_{ij}) + \frac{1}{h_{n+1}^5}\sum_{i,j,k=2}^{n} h_k^2 (h_i h_j h_{ij} - h_i^2 h_{jj})\\
&\quad + \frac{2x_{n+1}}{h_{n+1}^6}\sum_{i,j=2}^{n}(h_i h_j h_{n+1,i}h_{n+1,j} - h_i^2 h_{n+1,j}^2) + \frac{2x_{n+1}}{h_{n+1}^6}h_{n+1,n+1}\sum_{i,j=2}^{n}(h_i^2 h_{jj} - h_i h_j h_{ij})\\
&\quad + \frac{x_{n+1}}{h_{n+1}^6}\sum_{i,j,k=2}^{n} h_k^2 (h_i h_j h_{n+1,i}h_{n+1,j} - h_i^2 h_{n+1,j}^2) + \frac{x_{n+1}}{h_{n+1}^6}h_{n+1,n+1}\sum_{i,j,k=2}^{n} h_k^2 (h_i^2 h_{jj} - h_i h_j h_{ij})\\
&\quad + \frac{2x_{n+1}}{h_{n+1}^5}\sum_{i,j=2}^{n}(h_i h_{n+1,j}h_{ij} - h_i h_{n+1,i}h_{jj}) + \frac{x_{n+1}}{h_{n+1}^4}\sum_{i,j,k=2}^{n} h_i h_k (h_{jj}h_{ik} - h_{ij}h_{jk})
\end{aligned}$$

Likewise, the second-order derivatives of h are written in terms of derivatives of g.

$$
\begin{aligned}
h_{n+1,n+1} =& -\frac{1}{g_1^3} g_{11}, \\
h_{n+1,i} =& -\frac{1}{g_1^2} g_{1i} + \frac{1}{g_1^3} g_i g_{11} \text{ for } i = 2, \dots, n, \\
h_{ij} =& -\frac{1}{g_1} g_{ij} + \frac{1}{g_1^2} g_i g_{1j} + \frac{1}{g_1^2} g_j g_{1i} - \frac{1}{g_1^3} g_i g_j g_{11} \text{ for } i, j = 2, \dots, n.
\end{aligned} \tag{5.1.2}
$$

Using the notation $\mathcal{I} = x_{n+1}^2 + h_{n+1}^2 + x_{n+1}^2 \sum_{i=2}^n h_i^2 = h_{n+1}^2 I$, the the evolution of h is given by

$$
\begin{aligned}
\partial_t h =& \frac{1}{\mathcal{I}^{3/2}} (h_{n+1} - x_{n+1} h_{n+1,n+1}) \sum_{i=2}^n h_{ii} + \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} (h_{n+1} - x_{n+1} h_{n+1,n+1}) \sum_{i,j=2}^n (h_j^2 h_{ii} - h_i h_j h_{ij}) \\
& - \frac{1}{\sqrt{\mathcal{I}}} h_{n+1}^2 x_{n+1} \Big(-\frac{1}{h_{n+1}^4} \sum_{i,j=2}^n (h_j^2 h_{n+1,i}^2 - h_i h_j h_{n+1,i} h_{n+1,j}) - \frac{1}{h_{n+1}^4} \sum_{i=2}^n h_{n+1,i}^2 \Big) \\
& - \frac{1}{2\sqrt{\mathcal{I}}} h_{n+1}^2 x_{n+1} \Big(-\frac{4}{h_{n+1}^3} \sum_{i,j=2}^n (h_j h_{n+1,j} h_{ii} - h_i h_{n+1,j} h_{ij}) + \frac{1}{h_{n+1}^2} \sum_{i,j=2}^n (h_{ii} h_{jj} - h_{ij}^2) \Big) \\
& + \frac{1}{\mathcal{I}^{3/2}} h_{n+1}^4 x_{n+1}^2 \Big(-\frac{x_{n+1}}{h_{n+1}^6} \sum_{i=2}^n h_{n+1,i}^2 + \frac{x_{n+1}}{h_{n+1}^6} \sum_{i,j=2}^n h_i h_j (h_{ij} h_{n+1,n+1} - h_{n+1,i} h_{n+1,j}) \\
& - \frac{x_{n+1}}{h_{n+1}^5} \sum_{i,j,k=2}^n h_i h_k \big(h_{jj} (h_i h_{n+1,k} + h_k h_{n+1,i}) - h_{jk} (h_i h_{n+1,j} + h_j h_{n+1,i}) \big) \\
& - \frac{1}{h_{n+1}^5} \sum_{i,j=2}^n (h_i^2 h_{jj} - h_i h_j h_{ij}) + \frac{1}{h_{n+1}^5} \sum_{i,j,k=2}^n h_k^2 (h_i h_j h_{ij} - h_i^2 h_{jj}) \\
& + \frac{2x_{n+1}}{h_{n+1}^6} \sum_{i,j=2}^n (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) + \frac{x_{n+1}}{h_{n+1}^6} h_{n+1,n+1} \sum_{i,j=2}^n (h_i^2 h_{jj} - 2 h_i h_j h_{ij}) \\
& + \frac{x_{n+1}}{h_{n+1}^6} \sum_{i,j,k=2}^n h_k^2 (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) + \frac{x_{n+1}}{h_{n+1}^6} h_{n+1,n+1} \sum_{i,j,k=2}^n h_k^2 (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
& + \frac{2x_{n+1}}{h_{n+1}^5} \sum_{i,j=2}^n (h_i h_{n+1,j} h_{ij} - h_i h_{n+1,i} h_{jj}) + \frac{x_{n+1}}{h_{n+1}^4} \sum_{i,j,k=2}^n h_i h_k (h_{jj} h_{ik} - h_{ij} h_{jk}) \Big).
\end{aligned} \tag{5.1.3}
$$

The third derivatives of g can be expressed by the third derivatives of in the same manner. For $i, j, m = 2, \dots, n$, we have

$$
\begin{aligned}
g_{111} =& -\frac{1}{h_{n+1}^4} h_{n+1,n+1,n+1} + \frac{3}{h_{n+1}^5} h_{n+1,n+1}^2, \\
g_{11m} =& \frac{3}{h_{n+1}^4} h_{n+1,n+1} h_{n+1,m} - \frac{1}{h_{n+1}^3} h_{n+1,n+1,m} - \frac{3h_m}{h_{n+1}^5} h_{n+1,n+1} h_{n+1,n+1} + \frac{h_m}{h_{n+1}^4} h_{n+1,n+1,n+1},
\end{aligned}
$$

$$
\begin{aligned}
g_{1im} =& -\frac{3h_i}{h_{n+1}^4}h_{n+1,n+1}h_{n+1,m} + \frac{h_{im}}{h_{n+1}^3}h_{n+1,n+1} + \frac{h_i}{h_{n+1}^3}h_{n+1,n+1,m} + \frac{2}{h_{n+1}^3}h_{n+1,i}h_{n+1,m} - \frac{1}{h_{n+1}^2}h_{n+1,i,m} \\
&+ \frac{3h_ih_m}{h_{n+1}^5}h_{n+1,n+1}^2 - \frac{h_{n+1,i}h_m}{h_{n+1}^4}h_{n+1,n+1} - \frac{h_ih_m}{h_{n+1}^4}h_{n+1,n+1,n+1} - \frac{2h_m}{h_{n+1}^4}h_{n+1,i}h_{n+1,n+1} + \frac{h_m}{h_{n+1}^3}h_{n+1,n+1,i}, \\
g_{1mm} =& -\frac{3h_m}{h_{n+1}^4}h_{n+1,n+1}h_{n+1,m} + \frac{h_{mm}}{h_{n+1}^3}h_{n+1,n+1} + \frac{h_m}{h_{n+1}^3}h_{n+1,n+1,m} + \frac{2}{h_{n+1}^3}h_{n+1,m}h_{n+1,m} - \frac{1}{h_{n+1}^2}h_{n+1,m,m} \\
&+ \frac{3h_m^2}{h_{n+1}^5}h_{n+1,n+1}^2 - \frac{h_{n+1,m}h_m}{h_{n+1}^4}h_{n+1,n+1} - \frac{h_m^2}{h_{n+1}^4}h_{n+1,n+1,n+1} - \frac{2h_m}{h_{n+1}^4}h_{n+1,m}h_{n+1,n+1} + \frac{h_m}{h_{n+1}^3}h_{n+1,n+1,m}, \\
g_{iim} =& -\frac{3h_ih_i}{h_{n+1}^4}h_{n+1,n+1}h_{n+1,m} + \frac{h_i}{h_{n+1}^3}h_{n+1,n+1}h_{im} + \frac{h_i}{h_{n+1}^3}h_{n+1,n+1}h_{im} + \frac{h_ih_i}{h_{n+1}^3}h_{n+1,n+1,m} \\
&- \frac{4h_i}{h_{n+1}^3}h_{n+1,i}h_{n+1,m} + \frac{2}{h_{n+1}^2}h_{n+1,i}h_{im} + \frac{2h_i}{h_{n+1}^2}h_{n+1,i,m} + \frac{1}{h_{n+1}^2}h_{ii}h_{n+1,m} - \frac{1}{h_{n+1}}h_{iim} \\
&+ \frac{3h_ih_jh_m}{h_{n+1}^5}h_{n+1,n+1}^2 - \frac{h_jh_m}{h_{n+1}^4}h_{n+1,n+1}h_{n+1,i} - \frac{h_ih_m}{h_{n+1}^4}h_{n+1,n+1}h_{n+1,j} - \frac{h_ih_jh_m}{h_{n+1}^4}h_{n+1,n+1,n+1} \\
&+ \frac{4h_ih_m}{h_{n+1}^4}h_{n+1,j}h_{n+1,n+1} - \frac{2h_m}{h_{n+1}^3}h_{n+1,j}h_{n+1,i} - \frac{2h_ih_m}{h_{n+1}^3}h_{n+1,n+1,j} \\
&- \frac{h_m}{h_{n+1}^3}h_{ij}h_{n+1,n+1} + \frac{h_m}{h_{n+1}^2}h_{n+1,i,j}, \\
g_{ijm} =& -\frac{3h_ih_j}{h_{n+1}^4}h_{n+1,n+1}h_{n+1,m} + \frac{h_j}{h_{n+1}^3}h_{n+1,n+1}h_{im} + \frac{h_i}{h_{n+1}^3}h_{n+1,n+1}h_{jm} + \frac{h_ih_j}{h_{n+1}^3}h_{n+1,n+1,m} \\
&- \frac{2h_i}{h_{n+1}^3}h_{n+1,j}h_{n+1,m} + \frac{1}{h_{n+1}^2}h_{n+1,j}h_{im} + \frac{h_i}{h_{n+1}^2}h_{n+1,j,m} \\
&- \frac{2h_j}{h_{n+1}^3}h_{n+1,i}h_{n+1,m} + \frac{1}{h_{n+1}^2}h_{n+1,i}h_{jm} + \frac{h_j}{h_{n+1}^2}h_{n+1,i,m} + \frac{1}{h_{n+1}^2}h_{ij}h_{n+1,m} - \frac{1}{h_{n+1}}h_{ijm}, \\
&+ \frac{3h_ih_jh_m}{h_{n+1}^5}h_{n+1,n+1}^2 - \frac{h_jh_m}{h_{n+1}^4}h_{n+1,n+1}h_{n+1,i} - \frac{h_ih_m}{h_{n+1}^4}h_{n+1,n+1}h_{n+1,j} - \frac{h_ih_jh_m}{h_{n+1}^4}h_{n+1,n+1,n+1} \\
&+ \frac{2h_ih_m}{h_{n+1}^4}h_{n+1,j}h_{n+1,n+1} - \frac{h_m}{h_{n+1}^3}h_{n+1,j}h_{n+1,i} - \frac{h_ih_m}{h_{n+1}^3}h_{n+1,n+1,j} \\
&+ \frac{2h_jh_m}{h_{n+1}^4}h_{n+1,i}h_{n+1,n+1} - \frac{h_m}{h_{n+1}^3}h_{n+1,i}h_{n+1,j} - \frac{h_jh_m}{h_{n+1}^3}h_{n+1,n+1,i} \\
&- \frac{h_m}{h_{n+1}^3}h_{ij}h_{n+1,n+1} + \frac{h_m}{h_{n+1}^2}h_{n+1,i,j}.
\end{aligned}
$$

The evolution of $\widetilde{h}$, which is either h_t or h_i, $i = 2, \ldots, n$, is given by the operator $\mathcal{L}$ below.

$$
\partial_t \widetilde{h} = \mathcal{L}\widetilde{h} := -\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i,j=2}^{n}(h_i^2h_{jj} + h_j^2h_{ii} - 2h_ih_jh_{ij})\widetilde{h}_{n+1,n+1}
$$

$$
\begin{aligned}
&- \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i=2}^{n} h_{ii}\widetilde{h}_{n+1,n+1}\\
&+ \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^3\sum_{i=2}^{n} h_{ii}\widetilde{h}_{n+1,n+1}\\
&+ \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^3\sum_{i,j=2}^{n} h_ih_jh_{ij}\widetilde{h}_{n+1,n+1}\\
&+ \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^3\sum_{i,j,k=2}^{n} h_k^2(h_i^2h_{jj} - h_ih_jh_{ij})\widetilde{h}_{n+1,n+1}\\
&+ \frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^3\sum_{i,j=2}^{n}(h_i^2h_{jj} - h_ih_jh_{ij})\widetilde{h}_{n+1,n+1}\\
&+ \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i,j=2}^{n}(h_i^2h_{n+1,j}\widetilde{h}_{n+1,j} + h_j^2h_{n+1,i}\widetilde{h}_{n+1,i})\\
&- \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i,j=2}^{n}(h_ih_j\widetilde{h}_{n+1,i}h_{n+1,j} + h_ih_jh_{n+1,i}\widetilde{h}_{n+1,j})\\
&+ \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}h_{n+1}\sum_{i,j=2}^{n}(h_ih_{jj}\widetilde{h}_{n+1,i} + h_jh_{ii}\widetilde{h}_{n+1,j} - h_ih_{ij}\widetilde{h}_{n+1,j} - h_jh_{ij}\widetilde{h}_{n+1,i})\\
&+ \frac{2}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i=2}^{n} h_{n+1,i}\widetilde{h}_{n+1,i}\\
&- \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1}\sum_{i,j,k=2}^{n} h_ih_kh_{jj}(h_i\widetilde{h}_{n+1,k} + h_k\widetilde{h}_{n+1,i})\\
&+ \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1}\sum_{i,j,k=2}^{n} h_ih_kh_{jk}(h_i\widetilde{h}_{n+1,j} + h_j\widetilde{h}_{n+1,i})\\
&+ \frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1}\sum_{i,j=2}^{n} h_ih_{ij}\widetilde{h}_{n+1,j}\\
&- \frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1}\sum_{i,j=2}^{n} h_ih_{jj}\widetilde{h}_{n+1,i}\\
&- \frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}\sum_{i=2}^{n} h_{n+1,i}\widetilde{h}_{n+1,i}\\
&- \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}\sum_{i,j=2}^{n} h_ih_j(h_{n+1,j}\widetilde{h}_{n+1,i} + h_{n+1,i}\widetilde{h}_{n+1,j})\\
&+ \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}\sum_{i,j,k=2}^{n} h_k^2(h_ih_jh_{n+1,j}\widetilde{h}_{n+1,i} + h_ih_jh_{n+1,i}\widetilde{h}_{n+1,j})
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k^2 h_i^2 h_{n+1,j} \widetilde{h}_{n+1,j} \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} (h_i h_j h_{n+1,j} \widetilde{h}_{n+1,i} + h_i h_j h_{n+1,i} \widetilde{h}_{n+1,j}) \\
&- \frac{4}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} h_i^2 h_{n+1,j} \widetilde{h}_{n+1,j} \\
&- \frac{1}{2\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (h_i^2 \widetilde{h}_{jj} + h_j^2 \widetilde{h}_{ii} - 2 h_i h_j \widetilde{h}_{ij}) \\
&+ \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1} \sum_{i,j=2}^{n} (h_i h_{n+1,i} \widetilde{h}_{jj} + h_j h_{n+1,j} \widetilde{h}_{ii} - h_i h_{n+1,j} \widetilde{h}_{ij} - h_j h_{n+1,i} \widetilde{h}_{ij}) \\
&- \frac{1}{2\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1}^2 \sum_{i,j=2}^{n} (h_{jj} \widetilde{h}_{ii} + h_{ii} \widetilde{h}_{jj} - 2 h_{ij} \widetilde{h}_{ij}) \\
&+ \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}} \left(\sum_{i,j=2}^{n} (h_j^2 \widetilde{h}_{ii} - h_i h_j \widetilde{h}_{ij}) + \sum_{i=2}^{n} \widetilde{h}_{ii} \right) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i=2}^{n} \widetilde{h}_{ii} \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^{n} h_i^2 \widetilde{h}_{jj} \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^{n} (h_i^2 \widetilde{h}_{jj} - h_i h_j \widetilde{h}_{ij}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (h_i h_j \widetilde{h}_{ij} - h_i^2 \widetilde{h}_{jj}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1}^2 \sum_{i,j,k=2}^{n} h_i h_k (h_{ik} \widetilde{h}_{jj} + h_{jj} \widetilde{h_{ik}} - h_{jk} \widetilde{h}_{ij} - h_{ij} \widetilde{h_{jk}}) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} h_i h_k \Big((h_i h_{n+1,k} + h_k h_{n+1,i}) \widetilde{h}_{jj} - (h_i h_{n+1,j} + h_j h_{n+1,i}) \widetilde{h_{jk}} \Big) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i=2}^{n} \widetilde{h}_{ii} \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} h_i h_j \widetilde{h}_{ij} \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j,k=2}^{n} h_k^2 (h_i^2 \widetilde{h}_{jj} - h_i h_j \widetilde{h}_{ij})
\end{aligned}
$$

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\begin{aligned}
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (h_i^2 \widetilde{h}_{jj} - h_i h_j \widetilde{h}_{ij}) \\
&- \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1,n+1} \sum_{i=2}^{n} \widetilde{h}_{ii} \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j=2}^{n} h_i h_{n+1,j} \widetilde{h}_{ij} \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j=2}^{n} h_i h_{n+1,i} \widetilde{h}_{jj} \\
&- \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (h_i \widetilde{h}_i h_{jj} + h_j \widetilde{h}_j h_{ii} - \widetilde{h}_i h_j h_{ij} - h_i \widetilde{h}_j h_{ij}) \\
&+ \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} \sum_{i,j=2}^{n} (h_i \widetilde{h}_i h_{n+1,j}^2 + h_j \widetilde{h}_j h_{n+1,i}^2) \\
&- \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} \sum_{i,j=2}^{n} (\widetilde{h}_i h_j h_{n+1,i} h_{n+1,j} + h_i \widetilde{h}_j h_{n+1,i} h_{n+1,j}) \\
&+ \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} \widetilde{h}_{n+1} \sum_{i,j=2}^{n} (h_i h_{n+1,i} h_{jj} + h_j h_{n+1,j} h_{ii} - h_i h_{n+1,j} h_{ij} - h_j h_{n+1,i} h_{ij}) \\
&+ \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1} \sum_{i,j=2}^{n} (\widetilde{h}_i h_{n+1,i} h_{jj} + \widetilde{h}_j h_{n+1,j} h_{ii} - \widetilde{h}_i h_{n+1,j} h_{ij} - \widetilde{h}_j h_{n+1,i} h_{ij}) \\
&- \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1} \widetilde{h}_{n+1} \sum_{i,j=2}^{n} (h_{ii} h_{jj} - h_{ij}^2) \\
&+ \frac{1}{2\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1} (h_{n+1} \widetilde{h}_{n+1} + x_{n+1}^2 \sum_{k=2}^{n} h_k \widetilde{h}_k) h_{n+1,n+1} \sum_{i,j=2}^{n} (h_i^2 h_{jj} + h_j^2 h_{ii} - 2 h_i h_j h_{ij}) \\
&- \frac{1}{2\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1} (h_{n+1} \widetilde{h}_{n+1} + x_{n+1}^2 \sum_{k=2}^{n} h_k \widetilde{h}_k) \sum_{i,j=2}^{n} (h_i^2 h_{n+1,j}^2 + h_j^2 h_{n+1,i}^2 - 2 h_i h_j h_{n+1,i} h_{n+1,j}) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1} (h_{n+1} \widetilde{h}_{n+1} + x_{n+1}^2 \sum_{k=2}^{n} h_k \widetilde{h}_k) h_{n+1} \sum_{i,j=2}^{n} (h_i h_{n+1,i} h_{jj} + h_j h_{n+1,j} h_{ii}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1} (h_{n+1} \widetilde{h}_{n+1} + x_{n+1}^2 \sum_{k=2}^{n} h_k \widetilde{h}_k) h_{n+1} \sum_{i,j=2}^{n} (h_i h_{n+1,j} h_{ij} + h_j h_{n+1,i} h_{ij}) \\
&+ \frac{1}{2\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1} (h_{n+1} \widetilde{h}_{n+1} + x_{n+1}^2 \sum_{k=2}^{n} h_k \widetilde{h}_k) h_{n+1}^2 \sum_{i,j=2}^{n} (h_{ii} h_{jj} - h_{ij}^2) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1} (h_{n+1} \widetilde{h}_{n+1} + x_{n+1}^2 \sum_{j=2}^{n} h_j \widetilde{h}_j) \sum_{i=2}^{n} (h_{n+1,n+1} h_{ii} - h_{n+1,i}^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\sqrt{\mathcal{I}}}\frac{\widetilde{h}_{n+1}}{h_{n+1}^3}x_{n+1}h_{n+1,n+1}\sum_{i,j=2}^{n}(h_i^2h_{jj}+h_j^2h_{ii}-2h_ih_jh_{ij})\\
&-\frac{1}{\sqrt{\mathcal{I}}}\frac{\widetilde{h}_{n+1}}{h_{n+1}^3}x_{n+1}\sum_{i,j=2}^{n}(h_i^2h_{n+1,j}^2+h_j^2h_{n+1,i}^2-2h_ih_jh_{n+1,i}h_{n+1,j})\\
&-\frac{2}{\sqrt{\mathcal{I}}}\frac{\widetilde{h}_{n+1}}{h_{n+1}^3}x_{n+1}h_{n+1}\sum_{i,j=2}^{n}(h_ih_{n+1,i}h_{jj}+h_jh_{n+1,j}h_{ii}-h_ih_{n+1,j}h_{ij}-h_jh_{n+1,i}h_{ij})\\
&+\frac{1}{\sqrt{\mathcal{I}}}\frac{\widetilde{h}_{n+1}}{h_{n+1}^3}x_{n+1}h_{n+1}^2\sum_{i,j=2}^{n}(h_{ii}h_{jj}-h_{ij}^2)\\
&+\frac{2}{\sqrt{\mathcal{I}}}\frac{\widetilde{h}_{n+1}}{h_{n+1}^3}x_{n+1}\sum_{i=2}^{n}(h_{n+1,n+1}h_{ii}-h_{n+1,i}^2)\\
&-\frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}}(h_{n+1}\widetilde{h}_{n+1}+x_{n+1}^2\sum_{i=2}^{n}h_i\widetilde{h}_i)\left(\sum_{i,j=2}^{n}(h_j^2h_{ii}-h_ih_jh_{ij})+\sum_{i=2}^{n}h_{ii}\right)\\
&-\frac{1}{\sqrt{\mathcal{I}}}\frac{\widetilde{h}_{n+1}}{h_{n+1}^2}\left(\sum_{i,j=2}^{n}(h_j^2h_{ii}-h_ih_jh_{ij})+\sum_{i=2}^{n}h_{ii}\right)\\
&+\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}}\sum_{i,j=2}^{n}(+2h_j\widetilde{h}_jh_{ii}-\widetilde{h}_ih_jh_{ij}-h_i\widetilde{h}_jh_{ij})\\
&+\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h}_{n+1}+x_{n+1}^2\sum_{j=2}^{n}h_j\widetilde{h}_j)h_{n+1}\sum_{i=2}^{n}h_{ii}\\
&+\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h}_{n+1}+x_{n+1}^2\sum_{k=2}^{n}h_k\widetilde{h}_k)h_{n+1}\sum_{i,j=2}^{n}h_i^2h_{jj}\\
&+\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h}_{n+1}+x_{n+1}^2\sum_{k=2}^{n}h_k\widetilde{h}_k)h_{n+1}\sum_{i,j=2}^{n}(h_i^2h_{jj}-h_ih_jh_{ij})\\
&-\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h}_{n+1}+x_{n+1}^2\sum_{l=2}^{n}h_l\widetilde{h}_l)h_{n+1}\sum_{i,j,k=2}^{n}h_k^2(h_ih_jh_{ij}-h_i^2h_{jj})\\
&-\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h}_{n+1}+x_{n+1}^2\sum_{l=2}^{n}h_l\widetilde{h}_l)x_{n+1}h_{n+1}^2\sum_{i,j,k=2}^{n}h_ih_k(h_{jj}h_{ik}-h_{ij}h_{jk})\\
&+\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h}_{n+1}+x_{n+1}^2\sum_{l=2}^{n}h_l\widetilde{h}_l)x_{n+1}h_{n+1}\sum_{i,j,k=2}^{n}h_ih_kh_{jj}(h_ih_{n+1,k}+h_kh_{n+1,i})\\
&-\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h}_{n+1}+x_{n+1}^2\sum_{l=2}^{n}h_l\widetilde{h}_l)x_{n+1}h_{n+1}\sum_{i,j,k=2}^{n}h_ih_kh_{jk}(h_ih_{n+1,j}+h_jh_{n+1,i})\\
&-\frac{6}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h}_{n+1}+x_{n+1}^2\sum_{k=2}^{n}h_k\widetilde{h}_k)x_{n+1}h_{n+1}\sum_{i,j=2}^{n}(h_ih_{n+1,j}h_{ij}-h_ih_{n+1,i}h_{jj})
\end{aligned}
$$

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\begin{aligned}
&- \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h}_{n+1} + x_{n+1}^2 \sum_{j=2}^n h_j \widetilde{h}_j) x_{n+1} \sum_{i=2}^n (h_{ii} h_{n+1,n+1} - h_{n+1,i}^2) \\
&- \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h}_{n+1} + x_{n+1}^2 \sum_{k=2}^n h_k \widetilde{h}_k) x_{n+1} \sum_{i,j=2}^n h_i h_j (h_{ij} h_{n+1,n+1} - h_{n+1,i} h_{n+1,j}) \\
&- \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h}_{n+1} + x_{n+1}^2 \sum_{l=2}^n h_l \widetilde{h}_l) x_{n+1} \sum_{i,j,k=2}^n h_k^2 (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) \\
&- \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h}_{n+1} + x_{n+1}^2 \sum_{l=2}^n h_l \widetilde{h}_l) x_{n+1} h_{n+1,n+1} \sum_{i,j,k=2}^n h_k^2 (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&- \frac{6}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h}_{n+1} + x_{n+1}^2 \sum_{k=2}^n h_k \widetilde{h}_k) x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^n (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&- \frac{6}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h}_{n+1} + x_{n+1}^2 \sum_{k=2}^n h_k \widetilde{h}_k) x_{n+1} \sum_{i,j=2}^n (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 h_{n+1} \sum_{i=2}^n h_{ii} \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^n h_i^2 h_{jj} \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^n (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 h_{n+1} \sum_{i,j,k=2}^n h_k^2 (h_i h_j h_{ij} - h_i^2 h_{jj}) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} h_{n+1}^2 \sum_{i,j,k=2}^n h_i h_k (h_{jj} h_{ik} - h_{ij} h_{jk}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^n h_i h_k \Big(h_{jj} (h_i h_{n+1,k} + h_k h_{n+1,i}) - h_{jk} (h_i h_{n+1,j} + h_j h_{n+1,i}) \Big) \\
&- \frac{4}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j=2}^n (h_i h_{n+1,j} h_{ij} - h_i h_{n+1,i} h_{jj}) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} \sum_{i=2}^n (h_{ii} h_{n+1,n+1} - h_{n+1,i}^2) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^n h_i h_j (h_{ij} h_{n+1,n+1} - h_{n+1,i} h_{n+1,j}) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^n h_k^2 (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2)
\end{aligned}
$$

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\begin{aligned}
&- \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j,k=2}^{n} h_k^2 (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&- \frac{4}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&- \frac{4}{\mathcal{I}^{3/2}} \frac{\widetilde{h}_{n+1}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 \widetilde{h}_{n+1} \sum_{i=2}^{n} h_{ii} \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 \widetilde{h}_{n+1} \sum_{i,j=2}^{n} h_i^2 h_{jj} \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^{n} h_i \widetilde{h}_i h_{jj} \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 \widetilde{h}_{n+1} \sum_{i,j=2}^{n} (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^{n} (2 h_i \widetilde{h}_i h_{jj} - \widetilde{h}_i h_j h_{ij} - h_i \widetilde{h}_j h_{ij}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 \widetilde{h}_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (h_i h_j h_{ij} - h_i^2 h_{jj}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j,k=2}^{n} h_k \widetilde{h}_k (h_i h_j h_{ij} - h_i^2 h_{jj}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (\widetilde{h}_i h_j h_{ij} + h_i \widetilde{h}_j h_{ij} - 2 h_i \widetilde{h}_i h_{jj}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \widetilde{h}_{n+1} \sum_{i,j,k=2}^{n} h_i h_k (h_{jj} h_{ik} - h_{ij} h_{jk}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1}^2 \sum_{i,j,k=2}^{n} \widetilde{h}_i h_k (h_{jj} h_{ik} - h_{ij} h_{jk}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1}^2 \sum_{i,j,k=2}^{n} h_i \widetilde{h}_k (h_{jj} h_{ik} - h_{ij} h_{jk}) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \widetilde{h}_{n+1} \sum_{i,j,k=2}^{n} h_i h_k \Big(h_{jj} (h_i h_{n+1,k} + h_k h_{n+1,i}) - h_{jk} (h_i h_{n+1,j} + h_j h_{n+1,i}) \Big) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} \widetilde{h}_i h_k \Big(h_{jj} (h_i h_{n+1,k} + h_k h_{n+1,i}) - h_{jk} (h_i h_{n+1,j} + h_j h_{n+1,i}) \Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} h_i \widetilde{h}_k \Big(h_{jj} (h_i h_{n+1,k} + h_k h_{n+1,i}) - h_{jk} (h_i h_{n+1,j} + h_j h_{n+1,i}) \Big) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} h_i h_k h_{jj} (\widetilde{h}_i h_{n+1,k} + \widetilde{h}_k h_{n+1,i}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} h_i h_k h_{jk} (\widetilde{h}_i h_{n+1,j} + \widetilde{h}_j h_{n+1,i}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \widetilde{h}_{n+1} \sum_{i,j=2}^{n} (h_i h_{n+1,j} h_{ij} - h_i h_{n+1,i} h_{jj}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j=2}^{n} \widetilde{h}_i h_{n+1,j} h_{ij} \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j=2}^{n} \widetilde{h}_i h_{n+1,i} h_{jj} \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} \widetilde{h}_i h_j (h_{ij} h_{n+1,n+1} - h_{n+1,i} h_{n+1,j}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} h_i \widetilde{h}_j (h_{ij} h_{n+1,n+1} - h_{n+1,i} h_{n+1,j}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k \widetilde{h}_k (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k \widetilde{h}_k (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (\widetilde{h}_i h_j h_{n+1,i} h_{n+1,j} + h_i \widetilde{h}_j h_{n+1,i} h_{n+1,j}) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k^2 h_i \widetilde{h}_i h_{n+1,j}^2 \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j,k=2}^{n} h_k \widetilde{h}_k (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j,k=2}^{n} h_k^2 (2 h_i \widetilde{h}_i h_{jj} - \widetilde{h}_i h_j h_{ij} - h_i \widetilde{h}_j h_{ij}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (2 h_i \widetilde{h}_i h_{jj} - \widetilde{h}_i h_j h_{ij} - h_i \widetilde{h}_j h_{ij}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} (\widetilde{h}_i h_j h_{n+1,i} h_{n+1,j} + h_i \widetilde{h}_j h_{n+1,i} h_{n+1,j})
\end{aligned}
$$

$$- \frac{4}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} h_i \widetilde{h_i} h_{n+1,j}^2$$

through straightforward computations which are similar to those in the derivation of (3.1.5). Hence

$$\begin{aligned}
\partial_t \widetilde{h} = \mathcal{L}\widetilde{h} = {} & x_{n+1}\Big(- \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} \sum_{i=2}^{n} h_{ii} - \frac{1}{2\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} \sum_{i,j=2}^{n} (h_i^2 h_{jj} + h_j^2 h_{ii} - 2 h_i h_j h_{ij}) + O(x_{n+1}^2)\Big) \widetilde{h}_{n+1,n+1} \\
& + 2 \sum_{i=2}^{n} \sqrt{x_{n+1}} \Big(\frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} \sqrt{x_{n+1}} \Big(h_{n+1,i} + \sum_{j=2}^{n} (h_j^2 h_{n+1,i} - h_i h_j h_{n+1,j}) \Big) + O(\sqrt{x_{n+1}}) \Big) \widetilde{h}_{n+1,i} \\
& + \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} (h_{n+1} - x_{n+1} h_{n+1,n+1}) \Big(\sum_{i=2}^{n} \widetilde{h}_{ii} + \sum_{i,j=2}^{n} (h_j^2 \widetilde{h}_{ii} - h_i h_j \widetilde{h}_{ij}) \Big) + \sum_{i,j=2}^{n} O(\sqrt{x_{n+1}}) \widetilde{h}_{ij} \\
& - \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}} \Big(\sum_{i=2}^{n} h_{ii} + \sum_{i,j=2}^{n} (h_j^2 h_{ii} - h_i h_j h_{ij}) \Big) h_{n+1} \widetilde{h}_{n+1} - \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} \Big(\sum_{i=2}^{n} h_{ii} + \sum_{i,j=2}^{n} (h_j^2 h_{ii} - h_i h_j h_{ij}) \Big) \widetilde{h}_{n+1} \\
& + \frac{3}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}} x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (h_j^2 h_{ii} - h_i h_j h_{ij}) \widetilde{h}_{n+1} - \frac{3}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}} x_{n+1} \sum_{i,j=2}^{n} (h_j^2 h_{n+1,i}^2 - h_i h_j h_{n+1,i} h_{n+1,j}) \widetilde{h}_{n+1} \\
& + \frac{3}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}} x_{n+1} \sum_{i=2}^{n} (h_{n+1,n+1} h_{ii} - h_{n+1,i}^2) \widetilde{h}_{n+1} - \frac{4}{\mathcal{I}^{3/2}} x_{n+1} \sum_{i,j=2}^{n} (h_j h_{n+1,j} h_{ii} - h_i h_{n+1,j} h_{ij}) \widetilde{h}_{n+1} \\
& + \frac{1}{2\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1}^2 \sum_{i,j=2}^{n} (h_{ii} h_{jj} - h_{ij}^2) h_{n+1} \widetilde{h}_{n+1} + O(\sqrt{x_{n+1}}) \widetilde{h}_{n+1} \\
& + \frac{2}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}} \sum_{i,j=2}^{n} (h_i h_{jj} - h_j h_{ij}) \widetilde{h}_i - \frac{2}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (h_i h_{jj} - h_j h_{ij}) \widetilde{h}_i \\
& + \frac{2}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} \sum_{i,j=2}^{n} (h_i h_{n+1,j}^2 - h_j h_{n+1,i} h_{n+1,j}) \widetilde{h}_i + O(\sqrt{x_{n+1}}) \sum_{i=2}^{n} \widetilde{h}_i .
\end{aligned} \tag{5.1.4}$$

Lemma 5.1.1. *The $n \times n$ symmetric matrix $\widetilde{A} = (\widetilde{a_{ij}})_{i,j=n+1,2,\ldots,n}$ is given by*

$$\begin{aligned}
x_{n+1} \widetilde{a}_{n+1,n+1} &= (n+1, n+1)\text{-th entry of the diffusion coefficients of the equation } (5.1.4), \\
\sqrt{x_{n+1}} \widetilde{a}_{n+1,i} &= (n+1, i)\text{-th entry of the diffusion coefficients of the equation } (5.1.4), \text{ for } i = 2, \ldots, n, \\
\widetilde{a}_{ij} &= (i, j)\text{-th entry of the diffusion coefficients of the equation } (5.1.4), \text{ for } i, j = 2, \ldots, n.
\end{aligned}$$

b_{n+1} is the coefficient of the first derivative $\widetilde{h}_{n+1}$ in the equation (5.1.4)*, and b_i, $i = 2, \ldots, n$, is the coefficient of the first derivative $\widetilde{h}_i$, $i = 2, \ldots, n$, in the equation* (5.1.4)*. Then there exists $\eta, \lambda, \nu > 0$ depending only on the initial data and the constant ρ_0 such that $\lambda^{-1}|\xi|^2 \le \widetilde{A}\xi_i\xi_j \le \lambda|\xi|^2$ for any $\xi \in \mathbb{R}^n$, $|b_{n+1}| \le \lambda$, $|b_i| \le \lambda, i = 2, \ldots, n$, and $b_{n+1} \ge \nu$ on $\mathcal{B}_\eta(P)$.*

Proof. By the relations (5.1.1), $\widetilde{A} = (\widetilde{a_{ij}})$ are given in terms of diffusion coefficients $A = (a_{ij})$ in the equation (3.1.2) as the following. Since $x_{n+1}\widetilde{a}_{n+1,n+1} = \sum_{i,j=1}^{n} g_i g_j a_{ij}$, $\sqrt{x_{n+1}}\widetilde{a}_{n+1,i} =$

$\sum_{j=1}^{n} g_j a_{ij}$ for $i = 2, \dots, n$, $\widetilde{a}_{ij} = a_{ij}$ for $i, j = 2, \dots, n$, we have

$$\begin{aligned}
\widetilde{a}_{n+1,n+1} =& \frac{1}{I^{3/2}} \sum_{1\le i,j\le n} \Big((g_i^2 g_{jj} - g_i g_j g_{ij}) - g^2 g_i^2 (g_i^2 g_{jj} - g_i g_j g_{ij}) \Big) \\
\widetilde{a}_{n+1,i} =& \frac{1}{I^{3/2}} \sum_{j\neq i} \Big(\sqrt{g}(g_i g_{jj} - g_j g_{ij}) + g^{5/2} g_i (g_k^2 g_{jj} - g_j g_k g_{kj}) - g^{5/2} g_i (g_i^2 g_{jj} - g_i g_j g_{ij}) \\
& - g^{5/2} g_j \sum_k (g_k^2 g_{ij} + g_i g_j g_{kk} - g_i g_k g_{jk} - g_j g_k g_{ik}) \Big), \text{ for } i = 2, \dots, n, \\
\widetilde{a}_{ii} =& \frac{1}{I^{3/2}} \Big(\sum_{j\neq i} \big(g_j^2 + g g_{jj} + g^3 \sum_k (g_k^2 g_{jj} - g_j g_k g_{kj}) \big) - g^3 \sum_k (g_i^2 g_{kk} - g_i g_k g_{ik}) \Big), \text{ for } i = 2, \dots, n, \\
\widetilde{a}_{ij} =& - \frac{1}{I^{3/2}} \Big(g_i g_j + g g_{ij} + g^3 \sum_k (g_k^2 g_{ij} + g_i g_j g_{kk} - g_i g_k g_{jk} - g_j g_k g_{ik}) \Big), \text{ for } i, j = 2, \dots, n;\ i \neq j.
\end{aligned}$$

From the estimates of $|\nabla g|$, $\sum_{1\le i,j\le n}(g_i^2 g_{jj} - g_i g_j g_{ij})$, $R_{g,2} = \sum_{1\le i,j\le n}(g_{ij} g_{jj} - g_{ij}^2)$, we see that $\widetilde{A}$ is uniformly elliptic, i.e. $C^{-1}|\xi|^2 \le \widetilde{A}\xi_i\xi_j \le C|\xi|^2$ for any $\xi \in \mathbb{R}^n$. Because $b_{n+1} = \frac{1}{\sqrt{I}}\Big(g R_{g,2} + \sum_{i,j=1}^{n}(g_j^2 g_{ii} + g_i^2 g_{jj} - 2 g_i g_j g_{ii} g_{jj})\Big) + \frac{1}{2I^{3/2}} g R_{g,2} - \frac{1}{\sqrt{I}} g \sum_{i,j=2}^{n}(g_{ii} g_{jj} - g_{ij}^2) + \frac{2}{g_1\sqrt{I}} g \sum_{i,j=2}^{n}(g_i g_{1i} g_{jj} - g_i g_{1j} g_{ij}) + O(\sqrt{g})$ and $R_{g,2} \ge -C$, we see that $|b_{n+1}| \le C$ and $b_{n+1} \ge \nu > 0$ on $\mathcal{B}_\eta(P)$ for some constants $\eta, C, \nu > 0$. Finally, the coefficients b_i of $\widetilde{h}_i$, $i = 2, \dots, n$, are given by the bounded terms $b_i = \frac{2}{g_1^2\sqrt{I}}(g_1^2 + g g_{11}) \sum_{i,j=2}^{n}(g_i g_{jj} - g_j g_{ij}) + \frac{2}{g_1\sqrt{I}} \sum_{i,j=2}^{n}(g_j^2 g_{1i} - g_i g_j g_{1i}) - \frac{2g}{g_1^2\sqrt{I}} \sum_{i,j=2}^{n}(g_i g_{1j}^2 - g_j g_{1i} g_{1j}) + O(\sqrt{g})$. □

By Lemma 5.1.1, the equation (5.1.4) is of the form

$$\widetilde{h}_t = x_{n+1}\widetilde{a}_{n+1,n+1}\widetilde{h}_{n+1,n+1} + \sqrt{x_{n+1}} \sum_{i=2}^{n} \widetilde{a}_{n+1,i}\widetilde{h}_{n+1,i} + \sum_{i=2}^{n} \widetilde{a}_{ij}\widetilde{h}_{ij} + b_{n+1}\widetilde{h}_{n+1} + \sum_{i=2}^{n} b_i\widetilde{h}_i \tag{5.1.5}$$

with $(\widetilde{a}_{ij})$ being uniformly eliptic, $b_{n+1} > 0$ uniformly bounded and bounded below by some constant $\nu > 0$ such that $b_{n+1} \ge \nu > 0$, and b_i, $i = 2, \cdots, n$, uniformly bounded for sufficiently small x_{n+1}. As a result, with minor changes in the higher dimension to the line of proof of Theorem 3.1 in [6], we obtain the following Hölder regularity, which is an analogue of the Krylov-Safonov estimate [13].

Lemma 5.1.2. *There exist a number $0 < \alpha < 1$ and positive constants η and C, depending only on ρ_0 and the initial data, such that*

$$\|h_t\|_{C_s^\alpha(\mathcal{B}_{\frac{\eta}{2}})} + \sum_{i=2}^{n} \|h_i\|_{C_s^\alpha(\mathcal{B}_{\frac{\eta}{2}})} \le C \tag{5.1.6}$$

with respect to the metric ds^2 with the distance function s (1.6.2).

For $\widetilde{h} = h_{n+1}$, straightforward calculations show that,

$$
\begin{aligned}
\widetilde{h}_t = &-\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i,j=2}^{n}(h_i^2 h_{jj} + h_j^2 h_{ii} - 2h_i h_j h_{ij})\widetilde{h}_{n+1,n+1}\\
&- \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i=2}^{n} h_{ii}\widetilde{h}_{n+1,n+1}\\
&+ \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^3\sum_{i=2}^{n} h_{ii}\widetilde{h}_{n+1,n+1}\\
&+ \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^3\sum_{i,j=2}^{n} h_i h_j h_{ij}\widetilde{h}_{n+1,n+1}\\
&+ \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^3\sum_{i,j,k=2}^{n} h_k^2(h_i^2 h_{jj} - h_i h_j h_{ij})\widetilde{h}_{n+1,n+1}\\
&+ \frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^3\sum_{i,j=2}^{n}(h_i^2 h_{jj} - h_i h_j h_{ij})\widetilde{h}_{n+1,n+1}\\
&+ \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i,j=2}^{n}(+h_i^2 h_{n+1,j}\widetilde{h}_{n+1,j} + h_j^2 h_{n+1,i}\widetilde{h}_{n+1,i})\\
&- \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i,j=2}^{n}(+h_i h_j \widetilde{h_{n+1,i}} h_{n+1,j} + h_i h_j h_{n+1,i}\widetilde{h_{n+1,j}})\\
&+ \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}h_{n+1}\sum_{i,j=2}^{n}(h_i h_{jj}\widetilde{h_{n+1,i}} + h_j h_{ii}\widetilde{h_{n+1,j}} - h_i h_{ij}\widetilde{h_{n+1,j}} - h_j h_{ij}\widetilde{h_{n+1,i}})\\
&+ \frac{2}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i=2}^{n} h_{n+1,i}\widetilde{h_{n+1,i}}\\
&- \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2 x_{n+1}h_{n+1}\sum_{i,j,k=2}^{n} h_i h_k h_{jj}(+h_i\widetilde{h_{n+1,k}} + h_k\widetilde{h_{n+1,i}})\\
&+ \frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2 x_{n+1}h_{n+1}\sum_{i,j,k=2}^{n} h_i h_k h_{jk}(+h_i\widetilde{h_{n+1,j}} + h_j\widetilde{h_{n+1,i}})\\
&+ \frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2 x_{n+1}h_{n+1}\sum_{i,j=2}^{n} h_i h_{ij}\widetilde{h_{n+1,j}}\\
&- \frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2 x_{n+1}h_{n+1}\sum_{i,j=2}^{n} h_i h_{jj}\widetilde{h_{n+1,i}}\\
&- \frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2 x_{n+1}\sum_{i=2}^{n} h_{n+1,i}\widetilde{h_{n+1,i}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} h_i h_j (h_{n+1,j} \widetilde{h_{n+1,i}} + h_{n+1,i} \widetilde{h_{n+1,j}}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (h_i h_j h_{n+1,j} \widetilde{h_{n+1,i}} + h_i h_j h_{n+1,i} \widetilde{h_{n+1,j}}) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k^2 h_i^2 h_{n+1,j} \widetilde{h_{n+1,j}} \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} (+h_i h_j h_{n+1,j} \widetilde{h_{n+1,i}} + h_i h_j h_{n+1,i} \widetilde{h_{n+1,j}}) \\
&- \frac{4}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} h_i^2 h_{n+1,j} \widetilde{h_{n+1,j}} \\
&- \frac{1}{2\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (+h_i^2 \widetilde{h_{jj}} + h_j^2 \widetilde{h_{ii}} - 2 h_i h_j \widetilde{h_{ij}}) \\
&+ \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1} \sum_{i,j=2}^{n} (h_i h_{n+1,i} \widetilde{h_{jj}} + h_j h_{n+1,j} \widetilde{h_{ii}} - h_i h_{n+1,j} \widetilde{h_{ij}} - h_j h_{n+1,i} \widetilde{h_{ij}}) \\
&- \frac{1}{2\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}^2} x_{n+1} h_{n+1}^2 \sum_{i,j=2}^{n} (h_{jj} \widetilde{h_{ii}} + h_{ii} \widetilde{h_{jj}} - 2 h_{ij} \widetilde{h_{ij}}) \\
&+ \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}} \left(\sum_{i,j=2}^{n} (+h_j^2 \widetilde{h_{ii}} - h_i h_j \widetilde{h_{ij}}) + \sum_{i=2}^{n} \widetilde{h_{ii}} \right) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i=2}^{n} \widetilde{h_{ii}} \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^{n} h_i^2 \widetilde{h_{jj}} \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^{n} (+h_i^2 \widetilde{h_{jj}} - h_i h_j \widetilde{h_{ij}}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (+h_i h_j \widetilde{h_{ij}} - h_i^2 \widetilde{h_{jj}}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1}^2 \sum_{i,j,k=2}^{n} h_i h_k (h_{ik} \widetilde{h_{jj}} + h_{jj} \widetilde{h_{ik}} - h_{jk} \widetilde{h_{ij}} - h_{ij} \widetilde{h_{jk}}) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} h_i h_k \Big((h_i h_{n+1,k} + h_k h_{n+1,i}) \widetilde{h_{jj}} - (h_i h_{n+1,j} + h_j h_{n+1,i}) \widetilde{h_{jk}} \Big) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i=2}^{n} h_{n+1,n+1} \widetilde{h_{ii}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}\sum_{i,j=2}^{n}h_ih_jh_{n+1,n+1}\widetilde{h_{ij}}\\
&+\frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1,n+1}\sum_{i,j,k=2}^{n}h_k^2(+h_i^2\widetilde{h_{jj}}-h_ih_j\widetilde{h_{ij}})\\
&+\frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1,n+1}\sum_{i,j=2}^{n}(+h_i^2\widetilde{h_{jj}}-h_ih_j\widetilde{h_{ij}})\\
&-\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}h_{n+1,n+1}\sum_{i=2}^{n}\widetilde{h_{ii}}\\
&+\frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1}\sum_{i,j=2}^{n}h_ih_{n+1,j}\widetilde{h_{ij}}\\
&-\frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1}\sum_{i,j=2}^{n}h_ih_{n+1,i}\widetilde{h_{jj}}\\
&-\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}h_{n+1,n+1}\sum_{i,j=2}^{n}(h_i\widetilde{h_i}h_{jj}+h_j\widetilde{h_j}h_{ii}-\widetilde{h_i}h_jh_{ij}-h_i\widetilde{h_j}h_{ij})\\
&+\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i,j=2}^{n}(+h_i\widetilde{h_i}h_{n+1,j}^2+h_j\widetilde{h_j}h_{n+1,i}^2)\\
&-\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\sum_{i,j=2}^{n}(+\widetilde{h_i}h_jh_{n+1,i}h_{n+1,j}+h_i\widetilde{h_j}h_{n+1,i}h_{n+1,j})\\
&+\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}\widetilde{h_{n+1}}\sum_{i,j=2}^{n}(h_ih_{n+1,i}h_{jj}+h_jh_{n+1,j}h_{ii}-h_ih_{n+1,j}h_{ij}-h_jh_{n+1,i}h_{ij})\\
&+\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}h_{n+1}\sum_{i,j=2}^{n}(\widetilde{h_i}h_{n+1,i}h_{jj}+\widetilde{h_j}h_{n+1,j}h_{ii}-\widetilde{h_i}h_{n+1,j}h_{ij}-\widetilde{h_j}h_{n+1,i}h_{ij})\\
&-\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}x_{n+1}h_{n+1}\widetilde{h_{n+1}}\sum_{i,j=2}^{n}(h_{ii}h_{jj}-h_{ij}^2)\\
&+\frac{1}{2\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{k=2}^{n}h_k\widetilde{h_k})h_{n+1,n+1}\sum_{i,j=2}^{n}(h_i^2h_{jj}+h_j^2h_{ii}-2h_ih_jh_{ij})\\
&-\frac{1}{2\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{k=2}^{n}h_k\widetilde{h_k})\sum_{i,j=2}^{n}(h_i^2h_{n+1,j}^2+h_j^2h_{n+1,i}^2-2h_ih_jh_{n+1,i}h_{n+1,j})\\
&-\frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{k=2}^{n}h_k\widetilde{h_k})h_{n+1}\sum_{i,j=2}^{n}(h_ih_{n+1,i}h_{jj}+h_jh_{n+1,j}h_{ii})\\
&+\frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{k=2}^{n}h_k\widetilde{h_k})h_{n+1}\sum_{i,j=2}^{n}(h_ih_{n+1,j}h_{ij}+h_jh_{n+1,i}h_{ij})
\end{aligned}
$$

$$+ \frac{1}{2\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1} (h_{n+1}\widetilde{h_{n+1}} + x_{n+1}^2 \sum_{k=2}^{n} h_k \widetilde{h_k}) h_{n+1}^2 \sum_{i,j=2}^{n} (h_{ii}h_{jj} - h_{ij}^2)$$

$$+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1} (h_{n+1}\widetilde{h_{n+1}} + x_{n+1}^2 \sum_{j=2}^{n} h_j \widetilde{h_j}) \sum_{i=2}^{n} (h_{n+1,n+1}h_{ii} - h_{n+1,i}^2)$$

$$+ \frac{1}{\sqrt{\mathcal{I}}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (h_i^2 h_{jj} + h_j^2 h_{ii} - 2h_i h_j h_{ij})$$

$$- \frac{1}{\sqrt{\mathcal{I}}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1} \sum_{i,j=2}^{n} (h_i^2 h_{n+1,j}^2 + h_j^2 h_{n+1,i}^2 - 2h_i h_j h_{n+1,i} h_{n+1,j})$$

$$- \frac{2}{\sqrt{\mathcal{I}}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1} h_{n+1} \sum_{i,j=2}^{n} (h_i h_{n+1,i} h_{jj} + h_j h_{n+1,j} h_{ii} - h_i h_{n+1,j} h_{ij} - h_j h_{n+1,i} h_{ij})$$

$$+ \frac{1}{\sqrt{\mathcal{I}}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1} h_{n+1}^2 \sum_{i,j=2}^{n} (h_{ii}h_{jj} - h_{ij}^2)$$

$$+ \frac{2}{\sqrt{\mathcal{I}}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1} \sum_{i=2}^{n} (h_{n+1,n+1}h_{ii} - h_{n+1,i}^2)$$

$$- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}} (h_{n+1}\widetilde{h_{n+1}} + x_{n+1}^2 \sum_{i=2}^{n} h_i \widetilde{h_i}) \left(\sum_{i,j=2}^{n} (h_j^2 h_{ii} - h_i h_j h_{ij}) + \sum_{i=2}^{n} h_{ii} \right)$$

$$- \frac{1}{\sqrt{\mathcal{I}}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^2} \left(\sum_{i,j=2}^{n} (h_j^2 h_{ii} - h_i h_j h_{ij}) + \sum_{i=2}^{n} h_{ii} \right)$$

$$+ \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h_{n+1}} \sum_{i,j=2}^{n} (+2h_j \widetilde{h_j} h_{ii} - \widetilde{h_i} h_j h_{ij} - h_i \widetilde{h_j} h_{ij})$$

$$+ \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h_{n+1}} + x_{n+1}^2 \sum_{j=2}^{n} h_j \widetilde{h_j}) h_{n+1} \sum_{i=2}^{n} h_{ii}$$

$$+ \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h_{n+1}} + x_{n+1}^2 \sum_{k=2}^{n} h_k \widetilde{h_k}) h_{n+1} \sum_{i,j=2}^{n} h_i^2 h_{jj}$$

$$+ \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h_{n+1}} + x_{n+1}^2 \sum_{k=2}^{n} h_k \widetilde{h_k}) h_{n+1} \sum_{i,j=2}^{n} (h_i^2 h_{jj} - h_i h_j h_{ij})$$

$$- \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h_{n+1}} + x_{n+1}^2 \sum_{l=2}^{n} h_l \widetilde{h_l}) h_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (h_i h_j h_{ij} - h_i^2 h_{jj})$$

$$- \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h_{n+1}} + x_{n+1}^2 \sum_{l=2}^{n} h_l \widetilde{h_l}) x_{n+1} h_{n+1}^2 \sum_{i,j,k=2}^{n} h_i h_k (h_{jj} h_{ik} - h_{ij} h_{jk})$$

$$+ \frac{3}{\mathcal{I}^{5/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 (h_{n+1}\widetilde{h_{n+1}} + x_{n+1}^2 \sum_{l=2}^{n} h_l \widetilde{h_l}) x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} h_i h_k h_{jj} (h_i h_{n+1,k} + h_k h_{n+1,i})$$

$$
\begin{aligned}
&-\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{l=2}^n h_l\widetilde{h_l})x_{n+1}h_{n+1}\sum_{i,j,k=2}^n h_ih_kh_{jk}(h_ih_{n+1,j}+h_jh_{n+1,i})\\
&-\frac{6}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{k=2}^n h_k\widetilde{h_k})x_{n+1}h_{n+1}\sum_{i,j=2}^n(h_ih_{n+1,j}h_{ij}-h_ih_{n+1,i}h_{jj})\\
&-\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{j=2}^n h_j\widetilde{h_j})x_{n+1}\sum_{i=2}^n(h_{ii}h_{n+1,n+1}-h_{n+1,i}^2)\\
&-\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{k=2}^n h_k\widetilde{h_k})x_{n+1}\sum_{i,j=2}^n h_ih_j(h_{ij}h_{n+1,n+1}-h_{n+1,i}h_{n+1,j})\\
&-\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{l=2}^n h_l\widetilde{h_l})x_{n+1}\sum_{i,j,k=2}^n h_k^2(h_ih_jh_{n+1,i}h_{n+1,j}-h_i^2h_{n+1,j}^2)\\
&-\frac{3}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{l=2}^n h_l\widetilde{h_l})x_{n+1}h_{n+1,n+1}\sum_{i,j,k=2}^n h_k^2(h_i^2h_{jj}-h_ih_jh_{ij})\\
&-\frac{6}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{k=2}^n h_k\widetilde{h_k})x_{n+1}h_{n+1,n+1}\sum_{i,j=2}^n(h_i^2h_{jj}-h_ih_jh_{ij})\\
&-\frac{6}{\mathcal{I}^{5/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2(h_{n+1}\widetilde{h_{n+1}}+x_{n+1}^2\sum_{k=2}^n h_k\widetilde{h_k})x_{n+1}\sum_{i,j=2}^n(h_ih_jh_{n+1,i}h_{n+1,j}-h_i^2h_{n+1,j}^2)\\
&+\frac{2}{\mathcal{I}^{3/2}}\frac{\widetilde{h_{n+1}}}{h_{n+1}^3}x_{n+1}^2h_{n+1}\sum_{i=2}^n h_{ii}\\
&+\frac{2}{\mathcal{I}^{3/2}}\frac{\widetilde{h_{n+1}}}{h_{n+1}^3}x_{n+1}^2h_{n+1}\sum_{i,j=2}^n h_i^2h_{jj}\\
&+\frac{2}{\mathcal{I}^{3/2}}\frac{\widetilde{h_{n+1}}}{h_{n+1}^3}x_{n+1}^2h_{n+1}\sum_{i,j=2}^n(h_i^2h_{jj}-h_ih_jh_{ij})\\
&-\frac{2}{\mathcal{I}^{3/2}}\frac{\widetilde{h_{n+1}}}{h_{n+1}^3}x_{n+1}^2h_{n+1}\sum_{i,j,k=2}^n h_k^2(h_ih_jh_{ij}-h_i^2h_{jj})\\
&-\frac{2}{\mathcal{I}^{3/2}}\frac{\widetilde{h_{n+1}}}{h_{n+1}^3}x_{n+1}^2x_{n+1}h_{n+1}^2\sum_{i,j,k=2}^n h_ih_k(h_{jj}h_{ik}-h_{ij}h_{jk})\\
&+\frac{2}{\mathcal{I}^{3/2}}\frac{\widetilde{h_{n+1}}}{h_{n+1}^3}x_{n+1}^2x_{n+1}h_{n+1}\sum_{i,j,k=2}^n h_ih_k\Big(h_{jj}(h_ih_{n+1,k}+h_kh_{n+1,i})-h_{jk}(h_ih_{n+1,j}+h_jh_{n+1,i})\Big)\\
&-\frac{4}{\mathcal{I}^{3/2}}\frac{\widetilde{h_{n+1}}}{h_{n+1}^3}x_{n+1}^2x_{n+1}h_{n+1}\sum_{i,j=2}^n(h_ih_{n+1,j}h_{ij}-h_ih_{n+1,i}h_{jj})\\
&-\frac{2}{\mathcal{I}^{3/2}}\frac{\widetilde{h_{n+1}}}{h_{n+1}^3}x_{n+1}^2x_{n+1}\sum_{i=2}^n(h_{ii}h_{n+1,n+1}-h_{n+1,i}^2)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} h_i h_j (h_{ij} h_{n+1,n+1} - h_{n+1,i} h_{n+1,j}) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j,k=2}^{n} h_k^2 (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&- \frac{4}{\mathcal{I}^{3/2}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j=2}^{n} (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&- \frac{4}{\mathcal{I}^{3/2}} \frac{\widetilde{h_{n+1}}}{h_{n+1}^3} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 \widetilde{h_{n+1}} \sum_{i=2}^{n} h_{ii} \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 \widetilde{h_{n+1}} \sum_{i,j=2}^{n} h_i^2 h_{jj} \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^{n} h_i \widetilde{h_i} h_{jj} \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 \widetilde{h_{n+1}} \sum_{i,j=2}^{n} (h_i^2 h_{jj} - h_i h_j h_{ij}) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j=2}^{n} (+2 h_i \widetilde{h_i} h_{jj} - \widetilde{h_i} h_j h_{ij} - h_i \widetilde{h_j} h_{ij}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 \widetilde{h_{n+1}} \sum_{i,j,k=2}^{n} h_k^2 (h_i h_j h_{ij} - h_i^2 h_{jj}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j,k=2}^{n} h_k \widetilde{h_k} (h_i h_j h_{ij} - h_i^2 h_{jj}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 h_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (+\widetilde{h_i} h_j h_{ij} + h_i \widetilde{h_j} h_{ij} - 2 h_i \widetilde{h_i} h_{jj}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \widetilde{h_{n+1}} \sum_{i,j,k=2}^{n} h_i h_k (h_{jj} h_{ik} - h_{ij} h_{jk}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1}^2 \sum_{i,j,k=2}^{n} \widetilde{h_i} h_k (h_{jj} h_{ik} - h_{ij} h_{jk}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1}^2 \sum_{i,j,k=2}^{n} h_i \widetilde{h_k} (h_{jj} h_{ik} - h_{ij} h_{jk})
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \widetilde{h_{n+1}} \sum_{i,j,k=2}^{n} h_i h_k \Big(h_{jj}(h_i h_{n+1,k} + h_k h_{n+1,i}) - h_{jk}(h_i h_{n+1,j} + h_j h_{n+1,i}) \Big) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} \widetilde{h_i} h_k \Big(h_{jj}(h_i h_{n+1,k} + h_k h_{n+1,i}) - h_{jk}(h_i h_{n+1,j} + h_j h_{n+1,i}) \Big) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} h_i \widetilde{h_k} \Big(h_{jj}(h_i h_{n+1,k} + h_k h_{n+1,i}) - h_{jk}(h_i h_{n+1,j} + h_j h_{n+1,i}) \Big) \\
&- \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} h_i h_k h_{jj} (\widetilde{h_i} h_{n+1,k} + \widetilde{h_k} h_{n+1,i}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j,k=2}^{n} h_i h_k h_{jk} (\widetilde{h_i} h_{n+1,j} + \widetilde{h_j} h_{n+1,i}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \widetilde{h_{n+1}} \sum_{i,j=2}^{n} (h_i h_{n+1,j} h_{ij} - h_i h_{n+1,i} h_{jj}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j=2}^{n} \widetilde{h_i} h_{n+1,j} h_{ij} \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1} \sum_{i,j=2}^{n} \widetilde{h_i} h_{n+1,i} h_{jj} \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} \widetilde{h_i} h_j (h_{ij} h_{n+1,n+1} - h_{n+1,i} h_{n+1,j}) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j=2}^{n} h_i \widetilde{h_j} (h_{ij} h_{n+1,n+1} - h_{n+1,i} h_{n+1,j}) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k \widetilde{h_k} (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k \widetilde{h_k} (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}^2) \\
&+ \frac{1}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k^2 (\widetilde{h_i} h_j h_{n+1,i} h_{n+1,j} + h_i \widetilde{h_j} h_{n+1,i} h_{n+1,j}) \\
&- \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} \sum_{i,j,k=2}^{n} h_k^2 h_i \widetilde{h_i} h_{n+1,j}^2 \\
&+ \frac{2}{\mathcal{I}^{3/2}} \frac{1}{h_{n+1}^2} x_{n+1}^2 x_{n+1} h_{n+1,n+1} \sum_{i,j,k=2}^{n} h_k \widetilde{h_k} (h_i^2 h_{jj} - h_i h_j h_{ij})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1,n+1}\sum_{i,j,k=2}^{n}h_k^2(2h_i\widetilde{h_i}h_{jj}-\widetilde{h_i}h_jh_{ij}-h_i\widetilde{h_j}h_{ij})\\
&+\frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}h_{n+1,n+1}\sum_{i,j=2}^{n}(2h_i\widetilde{h_i}h_{jj}-\widetilde{h_i}h_jh_{ij}-h_i\widetilde{h_j}h_{ij})\\
&+\frac{2}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}\sum_{i,j=2}^{n}(\widetilde{h_i}h_jh_{n+1,i}h_{n+1,j}+h_i\widetilde{h_j}h_{n+1,i}h_{n+1,j})\\
&-\frac{4}{\mathcal{I}^{3/2}}\frac{1}{h_{n+1}^2}x_{n+1}^2x_{n+1}\sum_{i,j=2}^{n}h_i\widetilde{h_i}h_{n+1,j}^2\\
&-\frac{1}{2\sqrt{\mathcal{I}}}h_{n+1}^2\bigg(\frac{1}{h_{n+1}^4}h_{n+1,n+1}\sum_{i,j=2}^{n}(h_i^2h_{jj}+h_j^2h_{ii}-2h_ih_jh_{ij})\\
&\quad-\frac{1}{h_{n+1}^4}\sum_{i,j=2}^{n}(h_i^2h_{n+1,j}^2+h_j^2h_{n+1,i}^2-2h_ih_jh_{n+1,i}h_{n+1,j})\\
&\quad-\frac{2}{h_{n+1}^3}\sum_{i,j=2}^{n}(h_ih_{n+1,i}h_{jj}+h_jh_{n+1,j}h_{ii}-h_ih_{n+1,j}h_{ij}-h_jh_{n+1,i}h_{ij})+\frac{1}{h_{n+1}^2}\sum_{i,j=2}^{n}(h_{ii}h_{jj}-h_{ij}^2)\\
&\quad+\frac{2}{h_{n+1}^4}\sum_{i=2}^{n}(h_{n+1,n+1}h_{ii}-h_{n+1,i}^2)\bigg)\\
&+\frac{1}{\mathcal{I}^{3/2}}h_{n+1}^4\bigg(-\frac{2x_{n+1}}{h_{n+1}^5}\sum_{i=2}^{n}h_{ii}-\frac{2x_{n+1}}{h_{n+1}^5}\sum_{i,j=2}^{n}h_i^2h_{jj}\\
&\quad-\frac{2x_{n+1}}{h_{n+1}^5}\sum_{i,j=2}^{n}(h_i^2h_{jj}-h_ih_jh_{ij})+\frac{2x_{n+1}}{h_{n+1}^5}\sum_{i,j,k=2}^{n}h_k^2(h_ih_jh_{ij}-h_i^2h_{jj})\\
&\quad+\frac{3x_{n+1}^2}{h_{n+1}^6}\sum_{i=2}^{n}(h_{ii}h_{n+1,n+1}-h_{n+1,i}^2)+\frac{3x_{n+1}^2}{h_{n+1}^6}\sum_{i,j=2}^{n}h_ih_j(h_{ij}h_{n+1,n+1}-h_{n+1,i}h_{n+1,j})\\
&\quad-\frac{3x_{n+1}^2}{h_{n+1}^5}\sum_{i,j,k=2}^{n}h_ih_k\Big(h_{jj}(h_ih_{n+1,k}+h_kh_{n+1,i})-h_{jk}(h_ih_{n+1,j}+h_jh_{n+1,i})\Big)\\
&\quad+\frac{6x_{n+1}^2}{h_{n+1}^6}\sum_{i,j=2}^{n}(h_ih_jh_{n+1,i}h_{n+1,j}-h_i^2h_{n+1,j}^2)\\
&\quad+\frac{6x_{n+1}^2}{h_{n+1}^6}h_{n+1,n+1}\sum_{i,j=2}^{n}(h_i^2h_{jj}-h_ih_jh_{ij})\\
&\quad+\frac{3x_{n+1}^2}{h_{n+1}^6}\sum_{i,j,k=2}^{n}h_k^2(h_ih_jh_{n+1,i}h_{n+1,j}-h_i^2h_{n+1,j}^2)\\
&\quad+\frac{3x_{n+1}^2}{h_{n+1}^6}h_{n+1,n+1}\sum_{i,j,k=2}^{n}h_k^2(h_i^2h_{jj}-h_ih_jh_{ij})
\end{aligned}
$$

$$+\frac{6x_{n+1}^2}{h_{n+1}^5}\sum_{i,j=2}^{n}(h_ih_{n+1,j}h_{ij}-h_ih_{n+1,i}h_{jj})$$
$$+\frac{3x_{n+1}^2}{h_{n+1}^4}\sum_{i,j,k=2}^{n}h_ih_k(h_{jj}h_{ik}-h_{ij}h_{jk})\Bigg).$$

When we compare $\partial_t h_{n+1}$ to $\partial_t h_i,\ i=2,\dots,n,$ we have the additional terms in $\partial_t h_{n+1}$

$$-\frac{1}{2\sqrt{\mathcal{I}}}h_{n+1}^2\Bigg(\frac{1}{h_{n+1}^4}h_{n+1,n+1}\sum_{i,j=2}^{n}(h_i^2h_{jj}+h_j^2h_{ii}-2h_ih_jh_{ij})$$
$$-\frac{1}{h_{n+1}^4}\sum_{i,j=2}^{n}(h_i^2h_{n+1,j}^2+h_j^2h_{n+1,i}^2-2h_ih_jh_{n+1,i}h_{n+1,j})$$
$$-\frac{2}{h_{n+1}^3}\sum_{i,j=2}^{n}(h_ih_{n+1,i}h_{jj}+h_jh_{n+1,j}h_{ii}-h_ih_{n+1,j}h_{ij}-h_jh_{n+1,i}h_{ij})+\frac{1}{h_{n+1}^2}\sum_{i,j=2}^{n}(h_{ii}h_{jj}-h_{ij}^2)$$
$$+\frac{2}{h_{n+1}^4}\sum_{i=2}^{n}(h_{n+1,n+1}h_{ii}-h_{n+1,i}^2)\Bigg)$$
$$+\frac{1}{\mathcal{I}^{3/2}}h_{n+1}^4\Bigg(-\frac{2x_{n+1}}{h_{n+1}^5}\sum_{i=2}^{n}h_{ii}-\frac{2x_{n+1}}{h_{n+1}^5}\sum_{i,j=2}^{n}h_i^2h_{jj}$$
$$-\frac{2x_{n+1}}{h_{n+1}^5}\sum_{i,j=2}^{n}(h_i^2h_{jj}-h_ih_jh_{ij})+\frac{2x_{n+1}}{h_{n+1}^5}\sum_{i,j,k=2}^{n}h_k^2(h_ih_jh_{ij}-h_i^2h_{jj})$$
$$+\frac{3x_{n+1}^2}{h_{n+1}^6}\sum_{i=2}^{n}(h_{ii}h_{n+1,n+1}-h_{n+1,i}^2)+\frac{3x_{n+1}^2}{h_{n+1}^6}\sum_{i,j=2}^{n}h_ih_j(h_{ij}h_{n+1,n+1}-h_{n+1,i}h_{n+1,j})$$
$$-\frac{3x_{n+1}^2}{h_{n+1}^5}\sum_{i,j,k=2}^{n}h_ih_k\big(h_{jj}(h_ih_{n+1,k}+h_kh_{n+1,i})-h_{jk}(h_ih_{n+1,j}+h_jh_{n+1,i})\big)$$
$$+\frac{6x_{n+1}^2}{h_{n+1}^6}\sum_{i,j=2}^{n}(h_ih_jh_{n+1,i}h_{n+1,j}-h_i^2h_{n+1,j}^2)$$
$$+\frac{6x_{n+1}^2}{h_{n+1}^6}h_{n+1,n+1}\sum_{i,j=2}^{n}(h_i^2h_{jj}-h_ih_jh_{ij})$$
$$+\frac{3x_{n+1}^2}{h_{n+1}^6}\sum_{i,j,k=2}^{n}h_k^2(h_ih_jh_{n+1,i}h_{n+1,j}-h_i^2h_{n+1,j}^2)$$
$$+\frac{3x_{n+1}^2}{h_{n+1}^6}h_{n+1,n+1}\sum_{i,j,k=2}^{n}h_k^2(h_i^2h_{jj}-h_ih_jh_{ij})$$
$$+\frac{6x_{n+1}^2}{h_{n+1}^5}\sum_{i,j=2}^{n}(h_ih_{n+1,j}h_{ij}-h_ih_{n+1,i}h_{jj})$$

$$
\begin{aligned}
&+\frac{3x_{n+1}^2}{h_{n+1}^4}\sum_{i,j,k=2}^{n} h_ih_k(h_{jj}h_{ik}-h_{ij}h_{jk})\Bigg)\\
=&-\frac{1}{2\sqrt{\mathcal{I}}}h_{n+1}^2\Bigg(\frac{1}{h_{n+1}^4}h_{n+1,n+1}\sum_{i,j=2}^{n}(h_i^2h_{jj}+h_j^2h_{ii}-2h_ih_jh_{ij})\\
&-\frac{1}{h_{n+1}^4}\sum_{i,j=2}^{n}(h_i^2h_{n+1,j}^2+h_j^2h_{n+1,i}^2-2h_ih_jh_{n+1,i}h_{n+1,j})\\
&-\frac{2}{h_{n+1}^3}\sum_{i,j=2}^{n}(h_ih_{n+1,i}h_{jj}+h_jh_{n+1,j}h_{ii}-h_ih_{n+1,j}h_{ij}-h_jh_{n+1,i}h_{ij})+\frac{1}{h_{n+1}^2}\sum_{i,j=2}^{n}(h_{ii}h_{jj}-h_{ij}^2)\\
&+\frac{2}{h_{n+1}^4}\sum_{i=2}^{n}(h_{n+1,n+1}h_{ii}-h_{n+1,i}^2)\Bigg)+O(\sqrt{x_{n+1}}),
\end{aligned}
$$

which is expressed in terms of derivatives of g as below:

$$
\begin{aligned}
&-\frac{1}{2g_1\sqrt{I}}R_{g,2}+O(\sqrt{g})\\
=&-\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i=2}^{n}h_{ii}\widetilde{h}_{n+1}-\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i,j=2}^{n}(h_j^2h_{ii}-h_ih_jh_{ij})\widetilde{h}_{n+1}\\
&+\frac{2}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}}\sum_{i,j=2}^{n}(h_ih_{jj}-h_jh_{ij})\widetilde{h}_i-\frac{1}{2\sqrt{\mathcal{I}}}\sum_{i,j=2}^{n}(h_{ii}h_{jj}-h_{ij}^2)+\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i=2}^{n}h_{n+1,i}^2\\
&+\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i,j=2}^{n}(h_i^2h_{n+1,j}^2+h_j^2h_{n+1,i}^2-2h_ih_jh_{n+1,i}h_{n+1,j})+O(\sqrt{x_{n+1}})\\
=&-\frac{2}{\sqrt{\mathcal{I}}}\sum_{i,j=2}^{n}(g_ih_{jj}-g_jh_{ij})\widetilde{h}_i-\frac{2}{\sqrt{\mathcal{I}}}\frac{1}{g_1}\sum_{i,j=2}^{n}(g_ig_{jj}-g_jg_{ij})\widetilde{h}_i\\
&-\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{g_1^2}\sum_{i,j=2}^{n}(g_{ii}g_{jj}-g_{ij}^2)+\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{g_1^4}\sum_{i,j=2}^{n}(g_ig_{1j}-g_jg_{1i})^2\\
&+\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i=2}^{n}h_{n+1,i}^2+\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i,j=2}^{n}(h_i^2h_{n+1,j}^2+h_j^2h_{n+1,i}^2-2h_ih_jh_{n+1,i}h_{n+1,j})+O(\sqrt{x_{n+1}})\\
=&\frac{2}{\sqrt{\mathcal{I}}}\frac{1}{g_1^3}\sum_{i,j=2}^{n}(g_j^2g_{1i}-g_ig_jg_{1i})\widetilde{h}_i-\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{g_1^2}\sum_{i,j=2}^{n}(g_{ii}g_{jj}-g_{ij}^2)+\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{g_1^4}\sum_{i,j=2}^{n}(g_ig_{1j}-g_jg_{1i})^2\\
&+\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i=2}^{n}h_{n+1,i}^2+\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i,j=2}^{n}(h_i^2h_{n+1,j}^2+h_j^2h_{n+1,i}^2-2h_ih_jh_{n+1,i}h_{n+1,j})+O(\sqrt{x_{n+1}}).
\end{aligned}
$$

Hence we have

$$\begin{aligned}\widetilde{h}_t =&\mathcal{L}\widetilde{h} + \frac{2}{\sqrt{\mathcal{I}}}\frac{1}{g_1^3}\sum_{i,j=2}^{n}(g_j^2 g_{1i} - g_i g_j g_{1i})\widetilde{h}_i - \frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{g_1^2}\sum_{i,j=2}^{n}(g_{ii}g_{jj} - g_{ij}^2) + O(\sqrt{x_{n+1}})\\ &+ \frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{g_1^4}\sum_{i,j=2}^{n}(g_i g_{1j} - g_j g_{1i})^2 + \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i=2}^{n} h_{n+1,i}^2 + \frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}\sum_{i,j=2}^{n}(h_i h_{n+1,j} - h_j h_{n+1,i})^2\end{aligned} \tag{5.1.7}$$

Then both $\mathcal{L}\widetilde{h}$ (5.1.4) and $\widetilde{\mathcal{L}h}$ satisfy Lemma 5.1.1 for the same uniform constants $\eta, \lambda, \nu > 0$ and they are in the form (5.1.5) on $\mathcal{B}_\eta(P)$, where the operator $\widetilde{\mathcal{L}}$ is defined by

$$\widetilde{\mathcal{L}h} := \mathcal{L}\widetilde{h} + \frac{2}{\sqrt{\mathcal{I}}}\frac{1}{g_1^3}\sum_{i,j=2}^{n}(g_j^2 g_{1i} - g_i g_j g_{1i})\widetilde{h}_i. \tag{5.1.8}$$

Among the other terms in (5.1.7),

$$G_1 = -\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{g_1^2}\sum_{i,j=2}^{n}(g_{ii}g_{jj} - g_{ij}^2) + O(\sqrt{x_{n+1}}) \tag{5.1.9}$$

is uniformly bounded for small x_{n+1}. The problem is that we don't have a bound for the terms $h_{n+1,i}$, for $i = 2, \ldots, n$, whereas the terms $\sqrt{x_{n+1}}h_{n+1,i}$, for $i = 2, \ldots, n$, is bounded for small x_{n+1}. However, it is clear that the last line of the equation (5.1.7) is nonnegative. So $\widetilde{h}$ is a supersolution of the equation

$$\widetilde{h}_t \geq \widetilde{\mathcal{L}h} + G_1. \tag{5.1.10}$$

Lemma 5.1.3. *There exists a constant $c > 0$, depending only on ρ_0 and the initial data, such that*

$$\begin{aligned}&x_{n+1}\Big(-\sum_{i=2}^{n} h_{ii} - \sum_{i,j=2}^{n}(h_j^2 h_{ii} - h_i h_j h_{ij})\Big)h_{n+1,n+1}^2\\ &+ 2\sum_{i=2}^{n} x_{n+1}\Big(h_{n+1,i} + \sum_{j=2}^{n}(h_j^2 h_{n+1,i} - h_i h_j h_{n+1,j})\Big)h_{n+1,n+1}h_{n+1,i}\\ &+ \big(h_{n+1} - x_{n+1}h_{n+1,n+1}\big)\Big(\sum_{i=2}^{n} h_{n+1,i}^2 + \sum_{i,j=2}^{n}(h_j^2 h_{n+1,i}^2 - h_i h_j h_{n+1,i}h_{n+1,j})\Big)\\ \geq\ & c\Big(\sum_{i=2}^{n} h_{n+1,i}^2 + \sum_{i,j=2}^{n}(h_j^2 h_{n+1,i}^2 - h_i h_j h_{n+1,i}h_{n+1,j})\Big).\end{aligned} \tag{5.1.11}$$

Proof. There is a constant $0 < \lambda < 1$, depending only on ρ_0 and the initial data, as $x_{n+1} \to 0^+$

near the boundary point where we have chosen the coordinates,

$$
\begin{aligned}
&\Big(-\sum_{i=2}^{n} h_{ii}-\sum_{i,j=2}^{n}(h_j^2h_{ii}-h_ih_jh_{ij})\Big)(h_{n+1}-x_{n+1}h_{n+1,n+1})\\
&-x_{n+1}\Big(\sum_{i=2}^{n} h_{n+1,i}^2+\sum_{i,j=2}^{n}(h_j^2h_{n+1,i}^2-h_ih_jh_{n+1,i}h_{n+1,j})\Big)\\
=&\frac{1}{g_1^4}\Big(g_1^2\sum_{i=2}^{n} g_{ii}+g\sum_{i=2}^{n}(g_{11}g_{ii}-g_{1i}^2)\Big)+o(1)\geq\lambda^2,\ \text{and}\\
&\lambda\leq-\sum_{i=2}^{n} h_{ii}-\sum_{i,j=2}^{n}(h_j^2h_{ii}-h_ih_jh_{ij})=\frac{1}{g_1}\sum_{i=2}^{n} g_{ii}+o(1)\leq\lambda^{-1}.
\end{aligned}
$$

Indeed, as $x_{n+1}\to 0^+$ near the boundary point where we have chosen the coordinates,

$$
\begin{aligned}
&\Big(-\sum_{i=2}^{n} h_{ii}-\sum_{i,j=2}^{n}(h_j^2h_{ii}-h_ih_jh_{ij})\Big)(h_{n+1}-x_{n+1}h_{n+1,n+1})\\
&-x_{n+1}\Big(\sum_{i=2}^{n} h_{n+1,i}^2+\sum_{i,j=2}^{n}(h_j^2h_{n+1,i}^2-h_ih_jh_{n+1,i}h_{n+1,j})\Big)\\
=&\frac{1}{g_1^6}(g_1^2+gg_{11})\Big(\sum_{i=2}^{n}(g_1^2g_{ii}-2g_1g_ig_{1i}+g_i^2g_{11})\\
&+\frac{1}{g_1^2}\sum_{i,j=2}^{n}(g_j^2(g_1^2g_{ii}-2g_1g_ig_{1i}+g_i^2g_{11})+g_ig_j\big(-g_1^2g_{ij}+g_1g_ig_{1j}+g_1g_jg_{1i}-g_ig_jg_{11}\big))\Big)\\
&-\frac{1}{g_1^6}g\Big(\sum_{i=2}^{n}(-g_1g_{1i}+g_ig_{11})^2+\frac{1}{g_1^2}\sum_{i,j=2}^{n}(g_j^2(-g_1g_{1i}+g_ig_{11})^2-g_ig_j(-g_1g_{1i}+g_ig_{11})(-g_1g_{1j}+g_jg_{11}))\Big)\\
=&\frac{1}{g_1^4}(g_1^2+gg_{11})\sum_{i=2}^{n} g_{ii}-\frac{1}{g_1^4}g\sum_{i=2}^{n} g_{1i}^2+\frac{1}{g_1^6}(g_1^2+gg_{11})\sum_{i=2}^{n}(-2g_1g_ig_{1i}+g_i^2g_{11})\\
&+\frac{1}{g_1^8}(g_1^2+gg_{11})\sum_{i,j=2}^{n}(g_j^2(g_1^2g_{ii}-2g_1g_ig_{1i}+g_i^2g_{11})+g_ig_j\big(-g_1^2g_{ij}+g_1g_ig_{1j}+g_1g_jg_{1i}-g_ig_jg_{11}\big))\Big)\\
&-\frac{1}{g_1^6}g\Big(\sum_{i=2}^{n}(-2g_1g_ig_{11}g_{1i}+g_i^2g_{11}^2)\\
&+\frac{1}{g_1^2}\sum_{i,j=2}^{n}(g_j^2(-g_1g_{1i}+g_ig_{11})^2-g_ig_j(-g_1g_{1i}+g_ig_{11})(-g_1g_{1j}+g_jg_{11}))\Big)\\
=&\frac{1}{g_1^4}\Big(g_1^2\sum_{i=2}^{n} g_{ii}+g\sum_{i=2}^{n}(g_{11}g_{ii}-g_{1i}^2)\Big)+o(1).
\end{aligned}
$$

Then $h_{n+1} - x_{n+1}h_{n+1,n+1} \geq \lambda^3$ and

$$\begin{aligned}
&x_{n+1}^2\Big(\sum_{i=2}^{n} h_{n+1,i}^2 + \sum_{i,j=2}^{n}(h_j^2 h_{n+1,i}^2 - h_ih_jh_{n+1,i}h_{n+1,j})\Big)\\
&\leq x_{n+1}\Big(-\sum_{i=2}^{n} h_{ii} - \sum_{i,j=2}^{n}(h_j^2 h_{ii} - h_ih_jh_{ij})\Big)(h_{n+1} - x_{n+1}h_{n+1,n+1}) - x_{n+1}\lambda^2\\
&\leq x_{n+1}\Big(-\sum_{i=2}^{n} h_{ii} - \sum_{i,j=2}^{n}(h_j^2 h_{ii} - h_ih_jh_{ij})\Big)(h_{n+1} - x_{n+1}h_{n+1,n+1} - \lambda^3), \text{ and}\\
&x_{n+1}\sqrt{\sum_{i=2}^{n} h_{n+1,i}^2 + \sum_{i,j=2}^{n}(h_j^2 h_{n+1,i}^2 - h_ih_jh_{n+1,i}h_{n+1,j})}\\
&\geq -\sqrt{x_{n+1}\Big(-\sum_{i=2}^{n} h_{ii} - \sum_{i,j=2}^{n}(h_j^2 h_{ii} - h_ih_jh_{ij})\Big)(h_{n+1} - x_{n+1}h_{n+1,n+1} - \lambda^3)}.
\end{aligned} \tag{5.1.12}$$

Then in (5.1.11), the left hand side is

$$\begin{aligned}
LHS \geq &\Bigg(|h_{n+1,n+1}|\sqrt{x_{n+1}\Big(-\sum_{i=2}^{n} h_{ii} - \sum_{i,j=2}^{n}(h_j^2 h_{ii} - h_ih_jh_{ij})\Big)}\\
&- \sqrt{\Big(\sum_{i=2}^{n} h_{n+1,i}^2 + \sum_{i,j=2}^{n}(h_j^2 h_{n+1,i}^2 - h_ih_jh_{n+1,i}h_{n+1,j})\Big)(h_{n+1} - x_{n+1}h_{n+1,n+1} - \lambda^3)}\Bigg)^2\\
&- \Big(\sum_{i=2}^{n} h_{n+1,i}^2 + \sum_{i,j=2}^{n}(h_j^2 h_{n+1,i}^2 - h_ih_jh_{n+1,i}h_{n+1,j})\Big)(h_{n+1} - x_{n+1}h_{n+1,n+1} - \lambda^3)\\
&+ \Big(h_{n+1} - x_{n+1}h_{n+1,n+1}\Big)\Big(\sum_{i=2}^{n} h_{n+1,i}^2 + \sum_{i,j=2}^{n}(h_j^2 h_{n+1,i}^2 - h_ih_jh_{n+1,i}h_{n+1,j})\Big)\\
\geq &\lambda^3\Big(\sum_{i=2}^{n} h_{n+1,i}^2 + \sum_{i,j=2}^{n}(h_j^2 h_{n+1,i}^2 - h_ih_jh_{n+1,i}h_{n+1,j})\Big)
\end{aligned} \tag{5.1.13}$$

and hence (5.1.11) follows with $c = \lambda^3$. □

Lemma 5.1.4. *There exists a number $\theta > 1$, depending only on the a priori bounds, for which if $h_{n+1} - m > 0$ on $\mathcal{B}_\delta$, then $w := (h_{n+1} - m)^\theta$ is a subsolution of the equation*

$$\partial_t w \leq \widetilde{\mathcal{L}} w + G_2 \tag{5.1.14}$$

where G_2 is a bounded term.

Proof. For $\widetilde{h} = h_{n+1}$, the derivatives of w up to order 2 are given by

$$
\begin{aligned}
w_{n+1} &= \theta(h_{n+1}-m)^{\theta-1}\widetilde{h}_{n+1},\ w_i = \theta(h_{n+1}-m)^{\theta-1}\widetilde{h}_i \text{ for } i=2,\dots,n,\\
w_{n+1,n+1} &= \theta(h_{n+1}-m)^{\theta-1}\widetilde{h}_{n+1,n+1} + \theta(\theta-1)(h_{n+1}-m)^{\theta-2}\widetilde{h}^2_{n+1},\\
w_{n+1,i} &= \theta(h_{n+1}-m)^{\theta-1}\widetilde{h}_{n+1,i} + \theta(\theta-1)(h_{n+1}-m)^{\theta-2}\widetilde{h}_{n+1}\widetilde{h}_i \text{ for } i=2,\dots,n,\\
w_{ij} &= \theta(h_{n+1}-m)^{\theta-1}\widetilde{h}_{ij} + \theta(\theta-1)(h_{n+1}-m)^{\theta-2}\widetilde{h}_i\widetilde{h}_j \text{ for } i=2,\dots,n.
\end{aligned}
\tag{5.1.15}
$$

The function w evolves by

$$
\begin{aligned}
\partial_t w =& \widetilde{\mathcal{L}}w + \theta(h_{n+1}-m)^{\theta-1}\frac{1}{2\sqrt{\mathcal{I}}}\frac{1}{g_1^4}\sum_{i,j=2}^{n}\Big(-g_1^2(g_{ii}g_{jj}-g_{ij}^2) + (g_ig_{1j}-g_jg_{1i})^2\Big)\\
&+ \theta(h_{n+1}-m)^{\theta-1}O(\sqrt{x_{n+1}}) + G
\end{aligned}
\tag{5.1.16}
$$

where the group G of terms in (5.1.16) is given by

$$
\begin{aligned}
G :=& -\theta(\theta-1)(h_{n+1}-m)^{\theta-2}\Big(x_{n+1}\Big(-\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h^2_{n+1}}\sum_{i=2}^{n}h_{ii} - \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h^2_{n+1}}\sum_{i,j=2}^{n}(h_j^2h_{ii}-h_ih_jh_{ij}) + O(x^2_{n+1})\Big)\widetilde{h}^2_{n+1}\\
&+ 2\sum_{i=2}^{n}x_{n+1}\widetilde{h}_{n+1}\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h^2_{n+1}}\Big(\widetilde{h}_i + \sum_{j=2}^{n}(h_j^2\widetilde{h}_i - h_ih_j\widetilde{h}_j)\Big)\widetilde{h}_i + O(1)\sum_{i=2}^{n}x_{n+1}\widetilde{h}_{n+1}\widetilde{h}_i\\
&+ \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h^2_{n+1}}\Big(h_{n+1}-x_{n+1}h_{n+1,n+1}\Big)\Big(\sum_{i=2}^{n}\widetilde{h}_i^2 + \sum_{i,j=2}^{n}(h_j^2\widetilde{h}_i^2 - h_ih_j\widetilde{h}_i\widetilde{h}_j)\Big) + \sum_{i,j=2}^{n}O(\sqrt{x_{n+1}})\widetilde{h}_i\widetilde{h}_j\Big)\\
&+ \theta(h_{n+1}-m)^{\theta-1}\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h^2_{n+1}}\sum_{i=2}^{n}\widetilde{h}_i^2 + \theta(h_{n+1}-m)^{\theta-1}\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h^2_{n+1}}\sum_{i,j=2}^{n}(h_j^2\widetilde{h}_i^2 - h_ih_j\widetilde{h}_i\widetilde{h}_j).
\end{aligned}
\tag{5.1.17}
$$

For some uniform constant $c > 0$, by the inequality (5.1.11), G in (5.1.17) satisfies

$$
\begin{aligned}
G \le& \frac{1}{2}\theta(h_{n+1}-m)^{\theta-2}\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h^2_{n+1}}\sum_{i=2}^{n}h^2_{n+1,i}\Big((h_{n+1}-m) - c(\theta-1) + O(\sqrt{x_{n+1}})\Big)\\
&+ \theta(h_{n+1}-m)^{\theta-2}\frac{2}{\sqrt{\mathcal{I}}}\frac{1}{h^2_{n+1}}\sum_{i,j=2}^{n}h_j^2h^2_{n+1,i}\Big((h_{n+1}-m) - c(\theta-1)\Big)\\
&+ \frac{1}{2}\theta(h_{n+1}-m)^{\theta-2}\frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h^2_{n+1}}\sum_{i=2}^{n}\Big(h_{n+1,i} + O(1)\Big((h_{n+1}-m) - c(\theta-1)\Big)^{-1}\Big)^2\Big((h_{n+1}-m) - c(\theta-1)\Big)\\
&+ \theta(\theta-1)(h_{n+1}-m)^{\theta-2}|O(1)| + \theta(h_{n+1}-m)^{\theta-2}|O(1)|^2|(h_{n+1}-m) - c(\theta-1)|^{-1},
\end{aligned}
\tag{5.1.18}
$$

whose right-hand side and the group of terms $\theta(h_{n+1}-m)^{\theta-1}O(\sqrt{x_{n+1}})$ in (5.1.16) are both less than some uniform constant $C > 0$ if we take $\theta > 1$ sufficiently large. Hence,

$\partial_t w \leq \widetilde{\mathcal{L}}w + G_2$ where

$$G_2 := \theta(h_{n+1} - m)^{\theta-1} \frac{1}{2\sqrt{\mathcal{I}}} \frac{1}{g_1^4} \sum_{i,j=2}^{n} \Big(- g_1^2(g_{ii}g_{jj} - g_{ij}^2) + (g_i g_{1j} - g_j g_{1i})^2 \Big) + 2C \tag{5.1.19}$$

with G_2 being bounded, because $\sum_{i,j=2}^{n}(g_i g_{1j} - g_j g_{1i})^2 = o(1)$ as $x_{n+1} \to 0^+$ near the boundary point P where we have chosen the coordinates (1.6.1). □

Similarly, the function h_{n+1}^{θ} satisfies the following.

Lemma 5.1.5. *There exists a number $\theta > 1$, depending on the a priori bounds, for which $w := h_{n+1}^{\theta}$ is a subsolution of the equation*

$$\partial_t w \leq \widetilde{\mathcal{L}}w + G_3 \tag{5.1.20}$$

where G_3 is a bounded term.

Proof. The proof is the same as that of Lemma 5.1.4 with $h_{n+1} - m$ replaced by h_{n+1}. The result is straightforward with $G_3 = (G_2)_{m=0}$. □

Also, it holds that

Lemma 5.1.6. *There exists a number $\theta > 1$, depending only on the a priori bounds, so that for any constant M, $w := M^{\theta} - h_{n+1}^{\theta}$ is a supersolution of the equation*

$$\partial_t w \geq \widetilde{\mathcal{L}}w - G_3 \tag{5.1.21}$$

where G_3 is the bounded term from Lemma 5.1.5.

Proof. In Lemma 5.1.5 we have $\partial_t w = -\partial_t h_{n+1}^{\theta} \geq -\widetilde{\mathcal{L}}h_{n+1}^{\theta} - G_3 = \widetilde{\mathcal{L}}w - G_3$. □

Hence, we have another Hölder estimate (Lemma 5.9 in [7] for dimension two):

Lemma 5.1.7. *There exist a number $0 < \alpha < 1$ and positive constants η and C depending only on the initial data and ρ_0 such that*

$$\|h_{n+1}\|_{C_s^{\alpha}(\mathcal{B}_{\frac{\eta}{2}})} \leq C. \tag{5.1.22}$$

Proof. Since $\widetilde{h}$ is a supersolution of the equation (5.1.10), with the help of Lemma (5.1.4), (5.1.5) and (5.1.6), the usage of Theorem 3.6 and 3.7 from [6] for the proof of Lemma 5.9 in [7] shows that there exists a uniform constant $0 < \gamma < 1$ such that at any $Q \in \mathcal{B}_{\eta}(P)$ and $\mathcal{B}_{\rho}^{\gamma}(Q) \subset \mathcal{B}_{\eta}(P)$, it holds that

$$\operatorname{osc}_{\mathcal{B}^{\gamma}_{\frac{\rho}{4}}(Q)} \widetilde{h} \leq \gamma \operatorname{osc}_{\mathcal{B}^{\gamma}_{\rho}(Q)} \widetilde{h} + k(\rho) \tag{5.1.23}$$

for a certain $k = k(\rho) \geq 0$ satisfying $k(\rho) \leq C\rho^{\beta}$ for uniform positive constants C and $\beta < 1$. The Hölder continuity (5.1.22) comes from (5.1.23) by a standard iteration argument. □

5.2 $C_s^{2,\alpha}$ estimates

Now, we want to get the $C_s^{2,\alpha}$ regularity of h on $\mathcal{B}_\eta(P)$ from its $C_s^{1,\alpha}$ regularity and the classical regularity theory for strictly parabolic equations, as done for the Gauss curvature flow in [7].

We denote $\tilde{z} = 1 + z = 1 + x_{n+1}$. For $0 < \mu < 1$, let us denote by C_μ the parabolic cylinder $C_\mu = \{z^2 + |y|^2 \leq \mu^2, -\mu^2 \leq t \leq 0\}$. Let h^r be the dilated function of h at a point $(r^2, y_r, t_r) \in \mathcal{B}_\eta(P)$

$$h^r(z, y, t) = \frac{1}{r^2} h(r^2 + r^2 z, y_r + ry, t_r + r^2 t). \tag{5.2.1}$$

Then the equation (5.1.3) and the relations that, for $i, j = 2, \ldots, n$, at $(r^2 + r^2 z, y_r + ry, t_r + r^2 t)$,

$$\begin{aligned} &h_{n+1} = h^r_{n+1},\ h_i = r h^r_i,\ h_{n+1,n+1} = \frac{1}{r^2} h^r_{n+1,n+1},\ h_{n+1,i} = \frac{1}{r} h^r_{n+1,i},\ h_{ij} = h^r_{ij}, \\ &\partial_t h^r = h_{t|(r^2+r^2 z, y_r+ry, t_r+r^2 t)},\ \mathcal{I} = (h^r_{n+1})^2 + r^4 (\tilde{z})^2 + r^6 (\tilde{z})^2 \sum_{i=2}^n (h^r_i)^2 \end{aligned} \tag{5.2.2}$$

give the evolution of h^r, as follows.

$$\begin{aligned} \partial_t h^r = &-\frac{1}{\sqrt{\mathcal{I}}} \frac{\tilde{z}}{(h^r_{n+1})^2} \sum_{i=2}^n \left(h^r_{n+1,n+1} h^r_{ii} - (h^r_{n+1,i})^2 \right) + \frac{1}{\sqrt{\mathcal{I}}} \frac{1}{h^r_{n+1}} \sum_{i=2}^n h^r_{ii} + \frac{r^2}{\sqrt{\mathcal{I}}} \frac{1}{h^r_{n+1}} \sum_{i,j=2}^n \left((h^r_j)^2 h^r_{ii} - h^r_i h^r_j h^r_{ij} \right) \\ &- \frac{1}{2\sqrt{\mathcal{I}}} (h^r_{n+1})^2 r^2 \tilde{z} \Big(\frac{1}{(h^r_{n+1})^4} h^r_{n+1,n+1} \sum_{i,j=2}^n \left((h^r_i)^2 h^r_{jj} + (h^r_j)^2 h_{ii} - 2 h^r_i h^r_j h^r_{ij} \right) \\ &- \frac{1}{(h^r_{n+1})^4} \sum_{i,j=2}^n \left((h^r_i)^2 (h^r_{n+1,j})^2 + (h^r_j)^2 (h^r_{n+1,i})^2 - 2 h^r_i h^r_j h^r_{n+1,i} h^r_{n+1,j} \right) \\ &- \frac{2}{(h^r_{n+1})^3} \sum_{i,j=2}^n (h^r_i h^r_{n+1,i} h^r_{jj} + h^r_j h^r_{n+1,j} h^r_{ii} - h^r_i h^r_{n+1,j} h^r_{ij} - h^r_j h^r_{n+1,i} h^r_{ij}) + \frac{1}{(h^r_{n+1})^2} \sum_{i,j=2}^n \left(h^r_{ii} h^r_{jj} - (h^r_{ij})^2 \right) \Big) \\ &+ \frac{1}{\mathcal{I}^{3/2}} (h^r_{n+1})^4 r^4 (\tilde{z})^2 \Big(- \frac{1}{(h^r_{n+1})^5} \sum_{i=2}^n h^r_{ii} - \frac{1}{(h^r_{n+1})^5} r^2 \sum_{i,j=2}^n (h^r_j)^2 h^r_{ii} \\ &+ \frac{\tilde{z}}{(h^r_{n+1})^6} \sum_{i=2}^n \left(h^r_{ii} h^r_{n+1,n+1} - (h^r_{n+1,i})^2 \right) + \frac{r^2 \tilde{z}}{(h^r_{n+1})^6} \sum_{i,j=2}^n h^r_i h^r_j (h^r_{ij} h^r_{n+1,n+1} - h^r_{n+1,i} h^r_{n+1,j}) \\ &- \frac{r^2 \tilde{z}}{(h^r_{n+1})^5} r^2 \sum_{i,j,k=2}^n h^r_i h^r_k \Big(h^r_{jj} (h^r_i h^r_{n+1,k} + h^r_k h^r_{n+1,i}) - h^r_{jk} (h^r_i h^r_{n+1,j} + h^r_j h^r_{n+1,i}) \Big) \\ &- \frac{1}{(h^r_{n+1})^5} r^2 \sum_{i,j=2}^n \left((h^r_i)^2 h^r_{jj} - h^r_i h^r_j h^r_{ij} \right) + \frac{1}{(h^r_{n+1})^5} r^3 \sum_{i,j,k=2}^n (h^r_k)^2 \left(h^r_i h^r_j h^r_{ij} - (h^r_i)^2 h^r_{jj} \right) \\ &+ \frac{2 r^2 \tilde{z}}{(h^r_{n+1})^6} \sum_{i,j=2}^n \left(h^r_i h^r_j h^r_{n+1,i} h^r_{n+1,j} - (h^r_i)^2 (h^r_{n+1,j})^2 \right) + \frac{2 r^2 \tilde{z}}{(h^r_{n+1})^6} h^r_{n+1,n+1} \sum_{i,j=2}^n \left((h^r_i)^2 h^r_{jj} - h^r_i h^r_j h^r_{ij} \right) \end{aligned}$$

$$+\frac{r^2\tilde{z}}{(h^r_{n+1})^6}r^2\Big(\sum_{i,j,k=2}^{n}(h^r_k)^2\big(h^r_ih^r_jh^r_{n+1,i}h^r_{n+1,j}-(h^r_i)^2(h^r_{n+1,j})^2\big)+h^r_{n+1,n+1}\sum_{i,j,k=2}^{n}(h^r_k)^2\big((h^r_i)^2h^r_{jj}-h^r_ih^r_jh^r_{ij}\big)\Big)$$
$$+\frac{2r^2\tilde{z}}{(h^r_{n+1})^5}\sum_{i,j=2}^{n}(h^r_ih^r_{n+1,j}h^r_{ij}-h^r_ih^r_{n+1,i}h^r_{jj})+\frac{r^2\tilde{z}}{(h^r_{n+1})^4}r^2\sum_{i,j,k=2}^{n}h^r_ih^r_k(h^r_{jj}h^r_{ik}-h^r_{ij}h^r_{jk})\Big).$$

Lemma 5.2.1. *For any $0<\mu_0<1$, there exists a constant $C>0$ depending on μ_0, ρ_0 and the initial data such that*

$$\|h^r\|_{C^\infty(C_\mu)}\le C \tag{5.2.3}$$

for all $0<\mu<\mu_0$.

Proof. If $(z,y,t)\in C_\mu$ with $0<\mu<1$, then $\tilde{z}=1+z\ge 1-\mu^2>0$. By the bounds in Lemma (5.1.1) and the relation (5.2.2), the evolution of h^r is a uniformly parabolic equation on C_μ with the ellipticity constant λ_μ independent of r.

Hence, from the regularity of solutions to fully nonlinear uniformly parabolic equations (see Wang [16] and Wang [17]), $\|h^r\|_{C^\infty(C_\mu)}$ is, up to a uniform constant, bounded by $\|h^r\|_{L^\infty(C_{\mu_0})}$ for all $0<\mu<\mu_0<1$. Since h_{n+1} is bounded in $\mathcal{B}_\eta$, $\|h^r\|_{L^\infty(C_{\mu_0})}$ is uniformly bounded. □

Hence Lemma 6.4, Lemma 6.5 and Lemma 6.6 in Daskalopoulos and Lee [7] also hold for the transformed function h, and we get the following lemma.

Lemma 5.2.2. *There exist constants $0<\alpha<1$ and $C>0$, $\eta>0$, depending only on ρ_0 and the initial data, such that for any two points $P_1=(z_1,y_1,t_1)$ and $P_2=(z_2,y_2,t_2)$ in $\mathcal{B}_{\frac{\eta}{2}}$, we have*

$$|z_1h_{n+1,n+1}(P_1)-z_2h_{n+1,n+1}(P_2)|+\sum_{i=2}^{n}|\sqrt{z_1}h_{n+1,i}(P_1)-\sqrt{z_2}h_{n+1,i}(P_2)|\le Cs(P_1,P_2)^\alpha \tag{5.2.4}$$

In other words, $x_{n+1}h_{n+1,n+1}\in C^\alpha_s(\mathcal{B}_{\frac{\eta}{2}})$, $\sqrt{x_{n+1}}h_{n+1,i}\in C^\alpha_s(\mathcal{B}_{\frac{\eta}{2}})$ for $i=2,\dots,n$.

Lemma 5.2.3. *There exist constants $0<\alpha<1$ and $\eta>0$, depending only on ρ_0 and the initial data, such that*

$$\sum_{i=2}^{n}h_{ii},\sum_{i,j=2}^{n}h_ih_jh_{ij}\in C^\alpha_s(\mathcal{B}_{\frac{\eta}{2}})\text{ for }i,j=2,\dots,n. \tag{5.2.5}$$

Proof. In the equation (5.1.3), we have $h_t=\frac{1}{\mathcal{I}^{3/2}}(h_{n+1}-x_{n+1}h_{n+1,n+1})\sum_{i=2}^{n}h_{ii}+G_1$, where G_1 is a sum of bounded terms in $C^{\alpha'}_s(\mathcal{B}_{\frac{\eta}{2}})$, $\mathcal{I}$ bounded, $\mathcal{I}^{3/2}=h^3_{n+1}I^{3/2}\in C^{\alpha'}_s(\mathcal{B}_{\frac{\eta}{2}})$, for some $0<\alpha'<1$, by the Hölder regularity of h_{n+1}, h_t, h_i, $x_{n+1}h_{n+1,n+1}$ and $\sqrt{x_{n+1}}h_{n+1,i}$ for $i=2,\dots,n$ and boundedness of derivatives from Section 3 and 4 and the relations 5.1.1 and 5.1.2. Since $h_{n+1}-x_{n+1}h_{n+1,n+1}>c$ for some uniform constant $c>0$ on $\mathcal{B}_\eta$ and it belongs to $C^{\alpha'}_s(\mathcal{B}_{\frac{\eta}{2}})$, there exists a constant $0<\alpha<1$ such that

$$\sum_{i=2}^{n}h_{ii}=\frac{\mathcal{I}^{3/2}(h_t-G_1)}{h_{n+1}-x_{n+1}h_{n+1,n+1}}\in C^\alpha_s(\mathcal{B}_{\frac{\eta}{2}}). \tag{5.2.6}$$

So $\sum_{i,j=2}^{n} h_j^2 h_{ii} \in C_s^{\alpha}(\mathcal{B}_{\frac{\eta}{2}})$. Similarly, $h_t = \frac{1}{\sqrt{\mathcal{I}}}\frac{1}{h_{n+1}^2}(h_{n+1} - x_{n+1}h_{n+1,n+1})\sum_{i,j=2}^{n}(h_j^2 h_{ii} - h_i h_j h_{ij}) + G_2$, where G_2 is a sum of bounded terms in $C_s^{\alpha'}(\mathcal{B}_{\frac{\eta}{2}})$ and hence $\sum_{i,j=2}^{n}(h_j^2 h_{ii} - h_i h_j h_{ij}) \in C_s^{\alpha}(\mathcal{B}_{\frac{\eta}{2}})$. Consequently, $\sum_{i,j=2}^{n} h_i h_j h_{ij}$ belongs to $C_s^{\alpha}(\mathcal{B}_{\frac{\eta}{2}})$ as well. □

Lemma 5.2.4. *There exist constants $0 < \alpha < 1$ and $\eta > 0$, depending only on ρ_0 and the initial data, such that*

$$h_{ij} \in C_s^{\alpha}(\mathcal{B}_{\frac{\eta}{2}}) \text{ for } i, j = 2, \dots, n. \tag{5.2.7}$$

Proof. Since the tangential Laplacian $\sum_{i=2}^{n} h_{ii}$ belongs to $C_s^{\alpha}(\mathcal{B}_{\frac{\eta}{2}})$ by Lemma 5.2.3, the regularity of harmonic functions shows that (5.2.7) holds. □

In summary, we state Lemma 5.2.2 and Lemma 5.2.4 as the following.

Theorem 5.2.5. *Under the assumptions in the subsection 1.4, there exist uniform constants $0 < \alpha < 1$, $0 < C < \infty$ and $\eta > 0$ which depend only on the initial data and ρ_0, such that for any free boundary point $P = (0, y_0, t_0)$ with $0 < T_0 < t_0 < T$ satisfying $n_0 := \frac{P_0}{|P_0|} = e_1$, the function $x = h(z, y, t) = h(x_{n+1}, x_2, \dots, x_n, t)$ satisfies the Hölder estimate*

$$\|h\|_{C_s^{2+\alpha}(\mathcal{B}_\eta(P))} \leq C. \tag{5.2.8}$$

Chapter 6

All-time C^∞ regularity up to the interface

6.1 Proof of the main theorem

Let us show another lemma in order to prove the main theorem.

Lemma 6.1.1. *Let g be the solution which is smooth up to the interface in its support on $0 < t < T$ and $T < T_c$. Then there exist some constants $0 < \alpha < 1$ and $\lambda > 0$ depending only on ρ_0 and the initial data such that $g(\cdot, \tau)$ belongs to $C_s^{2+\alpha}(\{g \leq 1\})$ and satisfies the non-degenaracy conditions $|Dg(x, \tau)| \geq \lambda$ and $D^2_{\tau\tau} g(x, \tau) \geq \lambda$ for any $x \in \Gamma$ and $0 \leq \tau \leq T$.*

Proof. By the theorem 5.2.5 about the Hölder estimate of h with the relations (5.1.1) and (5.1.2) between the first-order and second-order derivatives of g and h, the conclusion is immediate. □

Finally, we prove our main theorem 1.7.1.

Proof. **Proof of Theorem 1.7.1.** By Theorem 1.6.2, there exists a solution g, smooth up to the interface in its support in $0 < t < T$, for a maximal time $T > T_0 > 0$. If $T < T_c$, then $g(\cdot, T)$ belongs to the class $C_s^{2+\alpha}(\{g \leq 1\})$ up to the interface $\Gamma(T)$ for some $0 < \alpha < 1$ and it satisfies the non-degenaracy conditions by Lemma 6.1.1. By Theorem 1.6.2, there exists a solution g in time $T \leq t \leq T + T'$ for some $T' > 0$ and it is smooth up to the interface in its support. This contradicts the condition that T is the maximal time. Hence, we must have $T = T_c$, the critical time of the flow. □

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국문초록

평평한 측면이 있는 콤팩트하고 볼록한 스칼라곡률흐름의 경계 근처에서의 행동에 대해 연구한다. 평평한 측면의 적당한 초기조건 아래에서, 평평한 측면이 사라질 때 까지 경계가 유한하고 정상적인 속도로 이동하는 것을 보인다. 그리고 압력 함수의 도함수의 최적 추정값, 레벨집합의 속도의 정상성, 경계 근처에서의 곡률의 최적 소멸 추정값, 그리고 Kim-Lee-Rhee의 곡률 하한값의 일반화된 버전을 얻고, 거기에서 자유경계까지의 최적 소멸률 대비 곡률의 비율의 헬더 정칙성을 획득한다. 마지막으로 지지집합에서 경계까지 매끄러운 해가 모든 시간 동안 존재하는 것을 보인다.

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